

SCHUBERT CLASSES OF A LOOP GROUP

PETER MAGYAR

In these notes, we survey the homology of the loop group ΩK of a compact group K , also known as the affine Grassmannian $\widehat{\text{Gr}}$ of a complex loop group $G[\mathbb{C}((t))]$. Using the Bott picture [B] of $H_*(\Omega K)$, the homology algebra or Pontryagin ring, we obtain two new results:

- A. Factorization of affine Schubert homology classes.
- B. Definition of affine Schubert polynomials representing the affine Schubert homology classes in all types, in terms similar to ordinary Schubert polynomials (Demazure operators on the Borel picture of the cohomology of the flag variety $H^*(K/T)$).

The philosophy is that working in the topological category (with compact rather than algebraic groups) gives extra flexibility which simplifies the constructions. For example, homology factorization happens because the Bott-Samelson variety is topologically a direct product, even though algebraically it is a non-trivial fiber bundle. Bott and Samelson exploit this flexibility to construct the Pontryagin ring $H_*(\Omega K)$ as the symmetric algebra of the homology $H_*(K/T)$. Once this construction is made, the computation of the Schubert classes becomes a standard exercise in the theory of Schubert polynomials in the equivariant cohomology $H_T^*(K/T)$. Thus, one may think of singular homology as a flabby version of equivariant K-theory and representation theory (Demazure modules). We work with rational (co)homology throughout this note, but with some care over denominators the results could be refined to integer coefficients.

In this note, for concreteness we work with $G = \text{SL}_n\mathbb{C}$, but unless otherwise noted everything generalizes to a simple, simply connected linear algebraic group of arbitrary type. In §1, we survey the relevant background. We give self-contained statements of Theorem A in §2.5 and Theorem B for $G = \text{SL}_n$ in §3.6. In an appendix §4, we discuss weak order and factorization in the affine Weyl group.

Our formulas are “global,” analogous to the Schubert polynomials in $H^*(G/B)$ and $H_T^*(G/B)$. Let us also mention the unpublished work of Dale Peterson [P], which contains a different and very powerful point of view on these topics based on the local picture of equivariant cohomology at T -fixed points (the nil-Hecke ring of Kostant-Kumar). Peterson shows that

the structure coefficients for $H_*(\Omega K)$ are certain Gromov-Witten invariants for the flag variety G/B , structure coefficients of the quantum cohomology ring $qH^*(G/B)$. An alternative proof of Peterson's theorem might well be possible by combining Bott's realization of $H_*(\Omega K)$ with the known structure of $qH^*(G/B)$.

Using Peterson's picture, Thomas Lam [L] has shown that one version of Lapointe & Morse's restricted Schur polynomials represent the Schubert classes of $H_*(\Omega K)$, so that these polynomials are identical to our Schubert polynomials for $G = \mathrm{SL}_n$. It should be possible to extend this theorem using the computation of affine Schubert polynomials in Theorem B.

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1. BACKGROUND ON TOPOLOGY AND COMBINATORICS

1.1. Loop group and affine Grassmannian. Let $G((t)) := \mathrm{SL}_n \mathbb{C}((t))$ be the group of determinant-one matrices with entries in the field of formal Laurent series $\mathbb{C}((t))$. Then $G((t))$ naturally includes the analytic functions $f : \mathbb{C}^\times \rightarrow G$, by identifying a function with its Laurent series. Let $G[[t]] := \mathrm{SL}_n \mathbb{C}[[t]]$ be the subgroup of determinant-one matrices with entries in the formal Taylor series $\mathbb{C}[[t]]$. We may regard $G[[t]]$ as a maximal parabolic subgroup of $G((t))$. The affine Grassmannian is the quotient space $\widehat{\mathrm{Gr}} := G((t))/G[[t]]$, which we can also realize as the $\mathbb{C}[[t]]$ -lattices with a fixed virtual dimension in the $\mathbb{C}((t))$ -vector space $\mathbb{C}((t))^n$.

Let K be a maximal compact subgroup of G , namely $K = \mathrm{SU}_n = \{A \in M_n(\mathbb{C}) \mid A\bar{A}^t = I\}$, the group of matrices whose column vectors form an oriented orthonormal basis for \mathbb{C}^n under the Hermitian inner product $\langle (a_1, \dots, a_n), (b_1, \dots, b_n) \rangle := \sum_i a_i \bar{b}_i$. Recall that $K \subset G$ is a maximal compact subgroup, a real Lie group which is not complex algebraic.

Let LK denote the group of real analytic loops¹ in K , the maps $f : S^1 \rightarrow K$, with multiplication performed pointwise: $(f \cdot g)(t) := f(t) \cdot g(t) \in K$. Considering $t \in S^1 \subset \mathbb{C}^\times$ as a complex parameter, any analytic loop $f(t) \in LK$ has a Laurent series, and we can consider $LK \subset G((t))$.

Let $\Omega K := \{f(t) \in LK \mid f(1) = 1_K\}$ be the normal subgroup of basepoint preserving loops. We can realize the affine Grassmannian as: $\widehat{\mathrm{Gr}} \cong \Omega K$. This holds because Gram-Schmidt orthogonalization allows us to write any $f(t) \in G((t))$ uniquely as $f(t) = f_K(t) \cdot f_P(t)$, where $f_K(t) \in \Omega K$ and $f_P(t) \in G[[t]]$, so $G((t)) = \Omega K \cdot G[[t]]$, a product of transverse subgroups. Thus the semi-infinite group ΩK provides a canonical slice (set of representatives) for the doubly infinite quotient $\widehat{\mathrm{Gr}} = G((t))/G[[t]]$. This is what makes the topological theory (using compact groups) simpler than the algebraic theory (using algebraic groups).

¹ More precisely, we mean the maps which extend to meromorphic functions $f : D \rightarrow K$ on the closed unit disk $D \subset \mathbb{C}$, with poles only allowed at $t = 0$.

It will sometimes be convenient to identify a basepoint-preserving loop in ΩK with the class of its translates by K . This leads to the realization of the affine Grassmannian as:

$$\widehat{\text{Gr}} \cong LK / (LK \cap G[[t]]) = LK/K \cong \Omega K.$$

where $K \subset LK$ is the group of constant loops $f(t) = k \in K$. Indeed, we can write any loop in LK as: $f(t) = f(t)f(1)^{-1} \cdot f(1) \in \Omega K \cdot K$, and this gives a natural homeomorphism $LK/K \cong \Omega K$. (This is not a group morphism, however.)

1.2. Pontryagin ring. Let us consider the singular homology of $\widehat{\text{Gr}} = \Omega K$. Although ΩK is infinite-dimensional, it is tamely infinite, meaning we can find an increasing union of finite-dimensional compact Hausdorff subspaces $\Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega K$ with $\Omega K = \bigcup_j \Omega_j$. (We can choose Ω_j from among the Schubert subvarieties we will define below.) Thus $H_*(\Omega K) = \varinjlim H_*(\Omega_j)$, a direct limit.²

We have seen that $\widehat{\text{Gr}} = \Omega K$ is topologically a loop group with a continuous (not complex algebraic) multiplication $\text{mult} : \Omega K \times \Omega K \rightarrow \Omega K$. This induces on the singular homology $H_*(\Omega K) := H_*(\Omega K, \mathbb{Q})$ a multiplication $\text{mult}_* : H_*(\Omega K) \otimes H_*(\Omega K) \rightarrow H_*(\Omega K)$, which together with the usual addition defines the so-called Pontryagin ring.

It is a basic fact that the Pontryagin multiplication is commutative. Furthermore, let us consider a different multiplication $\text{conc} : \Omega K \times \Omega K \rightarrow \Omega K$ defined by concatenating loops:

$$(f \tilde{\cdot} g)(e^{2\pi i x}) := \begin{cases} f(e^{4\pi i x}) & , \quad 0 \leq x \leq \frac{1}{2} \\ g(e^{4\pi i x}) & , \quad \frac{1}{2} \leq x \leq 1 \end{cases}.$$

Then the new induced multiplication conc_* on $H_*(\Omega K)$ is the *same* as mult_* .

Proof: We construct a homotopy of loops $f \tilde{\cdot} g \approx f \cdot g \approx g \tilde{\cdot} f$. For each $-1 \leq \epsilon \leq 1$, let $p_\epsilon : [0, 1] \rightarrow [0, 1] \times [0, 1]$ be a path in the unit square starting at $(0, 0)$ and ending at $(1, 1)$, such that p_{-1} goes along the left and top boundaries, p_0 goes along the diagonal, and p_1 goes along the bottom and right boundaries. Define $H : [0, 1] \times [0, 1] \rightarrow K$ by $H(x_1, x_2) := f(e^{2\pi i x_1})g(e^{2\pi i x_2})$. Then defining $h_\epsilon : S^1 \rightarrow K$ by $h_\epsilon(e^{2\pi i x}) := H(p_\epsilon(x))$ gives a continuous family of loops with $h_{-1} = f \tilde{\cdot} g$, $h_0 = f \cdot g$, and $h_1 = g \tilde{\cdot} f$. \square

As a corollary, the fundamental group of K (or of any Lie group) is abelian, since multiplying in $\pi_1(K)$ means concatenating loops. Of course, in our case $G = \text{SL}_n \mathbb{C}$ and $K = \text{SU}_n$ are simply connected, so the fundamental group is trivial (and the loop group ΩK is connected). On the other hand,

²Indeed, we can thicken each Ω_j into a normal neighborhood $U_j \subset \Omega K$ which is retractable to Ω_j . Since a singular cycle has compact support, it must live in a sufficiently large U_j , so that it is homologous to a cycle in Ω_j .

if we take the quotient of K by the center

$$Z = Z(G) = Z(K) = \left\{ e^{\frac{2\pi i}{n}j} I \mid j = 0, 1, \dots, n-1 \right\},$$

we get the adjoint groups $G^{\text{ad}} = G/Z$ and $K^{\text{ad}} = K/Z$. Since $G \rightarrow G^{\text{ad}}$ and $K \rightarrow K^{\text{ad}}$ are universal covering maps with fiber Z , we have $\pi_1(G^{\text{ad}}) \cong \pi_1(K^{\text{ad}}) \cong Z \cong \mathbb{Z}/n\mathbb{Z}$, a cyclic group of order n .

1.3. Coweight lattice and adjoint group. The subgroup of diagonal matrices $T \subset K$ is a maximal torus: $T \cong (S_1)^{n-1}$. Letting $\mathbb{R}^n = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_n$, we may realize the Lie algebra $\mathfrak{t} = \text{Lie}(T)$ as:

$$\mathfrak{t} \cong \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\}.$$

The exponential map is a group morphism and universal covering map: $\exp : \mathfrak{t} \rightarrow T$, $(x_1, \dots, x_n) \mapsto \text{diag}(e^{2\pi i x_1}, \dots, e^{2\pi i x_n})$. Its kernel is the coroot lattice:

$$\begin{aligned} Q^\vee &:= \text{Ker}(\exp : \mathfrak{t} \rightarrow T) \\ &= \{\lambda \in \mathbb{Z}^n \mid \lambda_1 + \dots + \lambda_n = 0\} = \bigoplus_{i=0}^{n-1} \mathbb{Z}\alpha_i^\vee, \end{aligned}$$

where $\alpha_i^\vee := e_i - e_{i+1}$ are the simple coroots. If we lift the center $Z = Z(G) = Z(K) \subset T$ up to \mathfrak{t} , we get the coweight lattice:

$$P^\vee := \exp^{-1}(Z) = \{\bar{\lambda} \mid \lambda \in \mathbb{Z}^n\} = \bigoplus_{i=1}^{n-1} \mathbb{Z}\varpi_i^\vee,$$

where we use the projection $- : \mathbb{R}^n \rightarrow \mathfrak{t}$, $\bar{\lambda} := \lambda - \frac{\lambda_1 + \dots + \lambda_n}{n}(e_1 + \dots + e_n)$, and $\varpi_i := \overline{e_1 + \dots + e_i} \in \frac{1}{n}\mathbb{Z}^n$ are the fundamental coweights. Exponentiating P^\vee back into T , we see that $Z \cong P^\vee / Q^\vee$.

Let us identify each vector $\lambda \in Q^\vee$ with the straight-line path $[0, 1] \rightarrow \mathfrak{t}$, $x \mapsto x\lambda$ and then exponentiate both sides. This makes $\lambda \in Q^\vee$ correspond to the loop $S^1 \rightarrow T$, $t \mapsto t^\lambda := \text{diag}(t^{\lambda_1}, \dots, t^{\lambda_n})$, and in fact $Q^\vee \cong \text{Hom}(S^1, T)$, the homomorphisms from the circle group to the torus. Similarly, $P^\vee \cong \text{Hom}(S^1, T^{\text{ad}})$, where $T^{\text{ad}} := T/Z$ is the maximal torus of $K^{\text{ad}} = K/Z$.

Consider ΩK^{ad} , the loops in the adjoint group. The connected components of ΩK^{ad} are by definition the elements of the fundamental group $\pi_1(K^{\text{ad}})$. Also, the identity component $\Omega K^{\text{ad}}_\circ$ (consisting of homotopically trivial loops) is normal, and the quotient is the component group $\Omega K^{\text{ad}} / \Omega K^{\text{ad}}_\circ \cong \pi_1(K^{\text{ad}})$.

We can understand ΩK^{ad} purely in terms of K . Every loop $h : [0, 1] \rightarrow K^{\text{ad}}$ with $h(0) = h(1) = 1$ lifts uniquely to a path $\tilde{h} : [0, 1] \rightarrow K$ starting at $\tilde{h}(0) = 1$ and ending at some $\tilde{h}(1) = z \in Z$. The map $\Omega K^{\text{ad}} \rightarrow Z$, $h(x) \mapsto \tilde{h}(1) = z$ realizes the quotient map to the component group $Z \cong \pi_1(K^{\text{ad}})$.

A homotopically trivial loop $h(x) \in \Omega K^{\text{ad}}_\circ$ lifts to a loop in K , meaning $\tilde{h}(1) = 1$ and $\tilde{h}(x) = f(e^{2\pi i x})$ for $f(t) \in \Omega K$, and this identifies $\Omega K^{\text{ad}}_\circ \cong \Omega K$.

As for a component corresponding to $z = \exp(\mu) \in Z$ with $\mu \in P^\vee$, the map $\Omega K \rightarrow \Omega K^{\text{ad}}$, $\tilde{h}(x) \mapsto \tilde{h}(x) \cdot \exp(2\pi i x \mu)$ (pointwise product) gives a homeomorphism from ΩK to the z -component of ΩK^{ad} .

1.4. Affine Weyl group. Let $W = S_n$ be the finite Weyl group, whose elements $w \in W$ act on $\mathfrak{t} \subset \mathbb{R}^n$ by permuting the n basis elements e_1, \dots, e_n , and on $\mathbb{R}^n \twoheadrightarrow \mathfrak{t}^*$ by permuting the dual basis elements x_1, \dots, x_n . Then W is a Coxeter group whose reflections correspond to the roots of G : for each pair of a coroot $\alpha^\vee = e_i - e_j \in \mathfrak{t}$ and its corresponding root $\alpha = x_i - x_j \in \mathfrak{t}^*$, we have the reflection $r_\alpha : \mathfrak{t} \rightarrow \mathfrak{t}$, $\mu \mapsto \mu - \langle \alpha, \mu \rangle \alpha^\vee$. The simple reflections are s_1, \dots, s_{n-1} for $\alpha_i = x_i - x_{i+1}$.

We also have the affine Weyl group $\widehat{W} = W \times Q^\vee$, a semi-direct product in which W acts on Q^\vee . For $w \in W$, $\lambda \in Q^\vee$, an element $wt^\lambda \in \widehat{W}$ acts on $\mathfrak{t} \subset \mathbb{R}^n$ by the affine transformation $\mu \mapsto w(\mu + \lambda)$, and $wt^\lambda = t^{w(\lambda)}w$. Then \widehat{W} is a Coxeter group whose (affine) reflections correspond to affine roots, pairs $\hat{\beta} = (\alpha, k)$ with α a root of W and $k \in \mathbb{Z}$: namely, $r_{\hat{\beta}} = r_{(\alpha, k)} := r_\alpha t^{k\alpha^\vee} = t^{-k\alpha^\vee} r_\alpha$.

The affine simple roots are $\hat{\alpha}_1 = (\alpha_1, 0)$, \dots , $\hat{\alpha}_{n-1} = (\alpha_{n-1}, 0)$, and $\hat{\alpha}_0 := (-\theta, 1)$, where $\theta = x_1 - x_n$ is the highest root and $\theta^\vee = e_1 - e_n$ its coroot.³ The affine simple reflections are $s_i := r_{\hat{\alpha}_i}$ for $i = 0, \dots, n-1$. Note that $s_1, \dots, s_{n-1} \in W \subset \widehat{W}$, and $s_0 = r_\theta t^{-\theta^\vee} = t^{\theta^\vee} r_\theta$.

We may consider the Weyl group as the quotient $W \cong N_K(T)/T$, where the normalizer $N_K(T)$ consists of scaled permutation matrices. We will denote a representative of $w \in W$ by $\overset{\circ}{w} \in N_K(T)$. Similarly, we have: $\widehat{W} \cong N_{LK}(L^0T)/L^0T$, where L^0T is the group of loops $f : S^1 \rightarrow T$ which extend to analytic functions on the closed unit disk $D \subset \mathbb{C}$. Furthermore, the loop $t^\lambda \in \text{Hom}(S^1, T) \subset N_{LK}(L^0T)$ gives a canonical representative for the corresponding element $t^\lambda \in \widehat{W}$, which justifies using the same symbol for both.

There is a technical unpleasantness in this definition of \widehat{W} , however. The adjoint action of $N_{LK}(L^0T)$ on \mathfrak{t} has kernel LT , not just L^0T , so the adjoint action does not produce the affine action of \widehat{W} on \mathfrak{t} . This can be remedied by passing to the central extension of LK (or of $G((t))$), which is the true affine Kac-Moody group.

1.5. Extended affine Weyl group. We also have the extended affine Weyl group $\widetilde{W} = W \times P^\vee$ consisting of affine transformations wt^λ with $w \in W$ and $\lambda \in P^\vee$, so that $\widetilde{W}/\widehat{W} \cong P^\vee/Q^\vee \cong Z$. There is a subgroup $\Sigma \subset \widetilde{W}$ such that $\widetilde{W} = \Sigma \cdot \widehat{W}$, a semi-direct product in which the subgroup $\Sigma \cong Z$ acts on \widehat{W} . In fact, $\Sigma = \text{Stab}_{\widetilde{W}}(A_0)$, where A_0 is the fundamental alcove, a simplex which is a fundamental domain of \widehat{W} acting on \mathfrak{t} :

$$A_0 := \{\lambda \in \mathfrak{t} \mid \langle \alpha_i, \lambda \rangle \geq 0, \langle \theta, \lambda \rangle \leq 1\}.$$

³Warning: For G not simply laced, θ^\vee is not the highest root of the dual root system.

We can explicitly describe the isomorphism $\Sigma \cong P^\vee/Q^\vee$ as follows:

$$\begin{aligned} \Sigma &\longleftrightarrow P^\vee/Q^\vee \\ \sigma &\longmapsto \eta = \sigma(0) \\ \sigma = t^\eta w_0 w_\eta &\longleftrightarrow \lambda, \end{aligned}$$

where η is the minimal dominant coweight with $\eta = \lambda \bmod Q^\vee$; and $w_0, w_\eta \in W$ are respectively the longest element and the longest element with $w(\eta) = \eta$. The η which occur give canonical representatives for P^\vee/Q^\vee : they are precisely the minuscule⁴ fundamental coweights $\eta = \varpi_i^\vee$ (and $\eta = 0$ for $\sigma = \text{id}$). We can lift σ to LK as: $\overset{\circ}{\sigma} := t^\eta \overset{\circ}{w}_0 \overset{\circ}{w}_\eta \in LK$.

Every $v \in \widetilde{W}$ has a unique decomposition $v = \sigma_v \widehat{v}$ with $\sigma_v \in \Sigma$ and $\widehat{v} \in \widehat{W}$. To extend the length function to \widetilde{W} we let $\ell(v) := \ell(\widehat{v})$, so that $\Sigma = \{v \in \widetilde{W} \mid \ell(v) = 0\}$.

1.6. Geometry of the center. We record some irrelevant facts about the geometry of the center Z and the adjoint group $K^{\text{ad}} = K/Z$.

Consider K^{ad} acting on K by conjugation. This clearly commutes with multiplication by a central element $z \in Z$ given by: $\tau : K \rightarrow K, k \mapsto z \cdot k$. (This is a deck-shuffling transformation of the universal covering map $K \rightarrow K^{\text{ad}}$.) Both of these actions are isometries of K (under a suitable Riemannian metric).

Since every conjugacy class of K intersects T in an orbit of the Weyl group W , we can construct the quotient as:

$$K/K^{\text{ad}} \cong T/W \cong \mathfrak{t}/(W \times Q^\vee) \cong A_0,$$

where A_0 is the fundamental alcove of the previous section. Now, $\tau = \tau_z$ induces an isometry of the quotient K/K^{ad} , namely:

$$\begin{aligned} \overline{\tau} : A_0 &\rightarrow A_0 \\ p &\mapsto p + \eta \bmod (W \times Q^\vee), \end{aligned}$$

where $\eta = \eta_z$ is the minimal dominant coweight with $\exp(2\pi i \eta) = z$: this is always a minuscule fundamental coweight, or $\eta = 0$ if $z = 1$. The formula $\sigma = t^\eta w_0 w_\eta$ lifts $\overline{\tau}$ to a map $\sigma : \mathfrak{t} \rightarrow \mathfrak{t}$ with $\sigma(A_0) = A_0$.

The geometry of A_0 is encoded in the affine Dynkin diagram $\widehat{\text{Dyn}}$: for example, the walls of A_0 correspond to the vertices of $\widehat{\text{Dyn}}$, the affine wall corresponding to the distinguished 0-node. A non-trivial isometry $\sigma = \sigma_z$ of A_0 defines a graph automorphism of $\widehat{\text{Dyn}}$ which moves the 0-node to the node i corresponding to the minuscule fundamental coweight $\varpi_i^\vee = \eta = \sigma(0)$. Thus we may embed $Z \hookrightarrow \text{Sym}(\widehat{\text{Dyn}})$. Finally, $\widehat{\text{Dyn}}$ is the Dynkin graph of the Kac-Moody group $G((t))$, and any element of $\widehat{\text{Dyn}}$ induces an outer automorphism of $G((t))$, LK , and ΩK .

⁴ A coweight $\lambda \in P^\vee$ is minuscule if $\langle \alpha, \lambda \rangle = 0$ or 1 for all positive roots α .

To summarize the many guises of the center:

$$\begin{aligned}
Z &= Z(G) = Z(K) \\
&\cong P^\vee / Q^\vee = \{0\} \cup \{\eta = \varpi_i^\vee \text{ minuscule}\} \\
&\cong \widehat{W} / \widehat{W} = \Sigma = \{\sigma \in W \times P^\vee \mid \sigma(A_0) = A_0\} \\
&\cong \pi_1(K^{\text{ad}}) \cong \Omega K^{\text{ad}} / \Omega K \\
&\hookrightarrow \text{Isometry}(A_0) \cong \text{Aut}(\widehat{\text{Dyn}}) \cong \text{Out } G((t)).
\end{aligned}$$

1.7. Flag varieties. Let $B \subset G$, the Borel subgroup of upper-triangular matrices in G . For a coweight $\lambda \in P^\vee$, we have the subgroup:

$$G^\lambda := \text{Stab}_G(t^\lambda) := \{g \in G \mid gt^\lambda g^{-1} = t^\lambda \text{ for all } t \in \mathbb{C}^\times\},$$

as well as the parabolic subgroup generated by these two subgroups: $P_\lambda := G_\lambda B \subset G$. The compact version is $K^\lambda := K \cap P^\lambda$. Note that $T \subset K^\lambda$ and $K^{w(\lambda)} = wK^\lambda w^{-1}$.

We will be concerned with $\varpi := \varpi_1^\vee = \overline{e_1} \in \mathfrak{t}$, the first fundamental coweight, for which we have:

$$P^\varpi = \{A = (a_{ij}) \in \text{SL}_n \mathbb{C} \mid a_{i1} = 0 \text{ for } j = 2, \dots, n\}$$

and $K^\varpi = \text{SU}_n \cap (\text{U}_1 \times \text{U}_{n-1})$, where $\text{U}_1 \times \text{U}_{n-1}$ denotes block-diagonal $n \times n$ unitary matrices. The corresponding partial flag variety is the complex projective $(n-1)$ -space:

$$X := G/P^\varpi = K/K^\varpi \cong \mathbb{P}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^\times = S^{2n-1} / S^1.$$

where we consider $S^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$.⁵ Explicitly, we have the homeomorphism:

$$\begin{aligned}
X = G/P^\varpi &\xrightarrow{\sim} \mathbb{P}^{n-1} = \mathbb{C}^n - \{0\} / \mathbb{C}^\times \\
A = (a_{ij}) &\mapsto A \cdot e_1 = [a_{11} : \dots : a_{n1}],
\end{aligned}$$

so that a matrix goes to the projective point given by its first column.

The full flag variety is the quotient space $\tilde{X} = G/B$, which can be realized as the space of all nested sequences of complex subspaces $(V_1 \subset V_2 \subset \dots \subset V_n = \mathbb{C}^n)$ with $\dim V_i = i$. The compact description of this is: $\tilde{X} \cong K/(K \cap B) = K/T$, which is the space of all orthonormal bases of \mathbb{C}^n modulo scaling each basis vector by S^1 : i.e., the flag variety is the space of all splittings of \mathbb{C}^n into orthogonal complex lines.

There is a right W -action on K/T : for $x = \overset{\circ}{x}T$ and $w = \overset{\circ}{w}T$, we let $x \cdot w := \overset{\circ}{x} \overset{\circ}{w}T$. This action is continuous but not holomorphic. For example, if $G = \text{SL}_2 \mathbb{C}$ and $w = (12) \in W$, we have $\overset{\circ}{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and for a typical flag

⁵ For $n = 2$, the circle bundle $S^1 \rightarrow S^3 \rightarrow \mathbb{P}^1 \cong S^2$ is known as the Hopf fibration.

$x = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \in G/B$ we compute:

$$\begin{array}{ccccccc} G/B & \cong & K/T & \xrightarrow{w} & K/T & \cong & G/B \\ x \bmod B & \mapsto & k \bmod T & \mapsto & kw \bmod T & \mapsto & y \bmod B \\ \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} & \mapsto & \begin{bmatrix} 1/d & -\bar{a}/d \\ a/d & 1/d \end{bmatrix} & \mapsto & \begin{bmatrix} -\bar{a}/d & -1/d \\ 1/d & -a/d \end{bmatrix} & \mapsto & \begin{bmatrix} 1 & 0 \\ -1/\bar{a} & 1 \end{bmatrix}, \end{array}$$

where $d := \sqrt{1 + a\bar{a}}$. Here the first map is Gram-Schmidt orthogonalization, the second map is the W -action, and the third map is the holomorphic coordinate function on a chart of $K/T \cong G/B \cong \mathbb{P}^1$. Thus, on this holomorphic chart the map is $w : a \mapsto -1/\bar{a}$, which is anti-holomorphic (and orientation-reversing).

1.8. Classifying spaces. Consider the direct limit vector space $\mathbb{C}^\infty := \lim_{j \rightarrow \infty} \mathbb{C}^j$ with the ℓ^2 metric and its unit sphere S^∞ , which is contractable since its homology is trivial, and which has a free right action of the circle group S^1 . The basic example of a classifying space is the quotient $B(S^1) = \mathbb{P}^\infty := S^\infty/S^1$, the (tamely) infinite projective space.

Consider $E := (S^\infty)_{\text{ind}}^n$, the space of linearly independent n -tuples of unit vectors $[\vec{v}_1, \dots, \vec{v}_n] \in (\mathbb{C}^\infty)^n$ thought of as $(\infty \times n)$ -matrices. This is contractable with a free right action of the torus $(S^1)^n$ and of the unitary group U_n , so the quotients $B(S^1)^n = E/(S^1)^n$ and $BU_n := E/U_n$ are classifying spaces of the respective groups. We have $B(S^1)^n \cong (\mathbb{P}^\infty)_{\text{ind}}^n$, the space of n -tuples of linearly independent lines in \mathbb{C}^∞ , and $BU_n \cong \text{Gr}(n, \mathbb{C}^\infty)$, the Grassmannian of n -planes in \mathbb{C}^∞ .

For a torus $T \subset (S^1)^n$, we define its classifying space as the free quotient $BT := E/T$. For any compact group with an n -dimensional representation, $K \subset U_n$, we let $BK := E/K$. (The space E serves as a contractable principal bundle for any such T and K , so that $E = ET = EK$.)

In our case $T \subset K = SU_n$, the space BT will be an S^1 -bundle over $B(S^1)^n$. Indeed, if we consider the dual tautological complex line bundle $\mathcal{D} := \mathcal{O}(1, \dots, 1)$ over $(\mathbb{P}^\infty)_{\text{ind}}^n$, or simply over $(\mathbb{P}^\infty)^n$, we can construct our space BT as the associated S^1 -bundle (Thom bundle) of the complex line bundle \mathcal{D} . An element of BT is essentially an n -tuple of complex lines $(\mathbb{C}\vec{v}_1, \dots, \mathbb{C}\vec{v}_n) \in (\mathbb{P}^\infty)^n$, together with a non-zero volume form⁶ on the complex n -space $V = \mathbb{C}\vec{v}_1 + \dots + \mathbb{C}\vec{v}_n$. We can also construct BK as the S^1 -bundle of the complex line bundle $\mathcal{D} = \mathcal{O}(1) = \wedge^n \text{Taut}^* \rightarrow \text{Gr}(n, \mathbb{C}^\infty) \cong BU_n$: that is, an element of BK is a complex n -space $V \subset \mathbb{C}^\infty$ endowed with a non-zero volume form.

We have the T -equivariant cohomology ring $H_T^*(\text{pt}) := H^*(BT) \cong \text{Sym}(\mathfrak{t}^*)$, the ring of polynomial functions on $\mathfrak{t} \subset \mathbb{R}^n$. Indeed, for an element of the coroot lattice $\lambda \in Q^\vee \subset \mathfrak{t}^*$, we have the one-dimensional character $\exp(\lambda) : T \rightarrow S^1$ and the associated complex line bundle $\mathcal{L}_\lambda := ET \times^T \mathbb{C}_{\exp(\lambda)} \rightarrow$

⁶ More precisely, a \mathbb{C} -linear n -form on V which agrees in absolute value with the standard form.

BT . Then λ is identified with the first Chern class $c_1(\mathcal{L}_\lambda) \in H^*(BT)$, the Poincare dual of the vanishing locus of a section of \mathcal{L} . In our case we have:

$$H_T^*(\text{pt}) = S := \mathbb{Q}[y'_1, \dots, y'_n] / (y'_1 + \dots + y'_n),$$

where y'_1, \dots, y'_n in $(\mathbb{R}^n)^*$ forms a dual basis of $e_1, \dots, e_n \in \mathbb{R}^n$. For historical and combinatorial reasons, we re-index the y -variables, letting:

$$y_i := y'_{n+1-i} = y'_{w_0(i)}.$$

We define a grading on $\mathbb{Q}[y]$ by writing: $\dim_{\mathbb{R}}(y_i) = 2$.

The space BK has the same rational cohomology (but *not* the same integer cohomology) as the finite quotient BT/W , where W permutes the entries of $(\vec{v}_1, \dots, \vec{v}_n) \in (\mathbb{P}^\infty)^n$: that is, $H^*(BK) \cong H_T^*(\text{pt})^W \cong S^W$, the W -invariant polynomials in S .

1.9. Cohomology of the flag variety. For a variety Y endowed with a T -action, we define the equivariant cohomology ring $H_T^*(Y) := H^*(Y_T)$, the ordinary cohomology of the tamely infinite-dimensional induced space:

$$Y_T := ET \overset{T}{\times} Y = (ET \times Y) / T,$$

where the free T -action on $ET \times Y$ is given by: $(e, y) \cdot t = (et, t^{-1}y)$.

Consider the flag variety $\tilde{X} = K/T$, with its T -equivariant cohomology ring $H_T^*(\tilde{X}) := H^*(\tilde{X}_T)$. We have a homeomorphism:

$$\begin{aligned} \tilde{X}_T = ET \overset{T}{\times} K/T &\xrightarrow{\sim} BT \underset{BK}{\times} BT \\ ([\vec{v}_1, \dots, \vec{v}_n], k) &\longmapsto ([\vec{v}_1, \dots, \vec{v}_n] \cdot k, [\vec{v}_1, \dots, \vec{v}_n]) \bmod T^2. \end{aligned}$$

From this it is easy to see that $H_T^*(\tilde{X}) \cong S \otimes_{S^W} S$.

We can realize the ordinary cohomology $H^*(\tilde{X})$ as a quotient of the equivariant cohomology $H_T^*(\tilde{X})$ as follows. Choose an arbitrary basepoint $b_\circ \in BT$, and consider the fiber bundle:

$$\tilde{X} \xrightarrow{\text{fib}} ET \overset{T}{\times} \tilde{X} \xrightarrow{\text{proj}} BT,$$

where $\text{fib} : \tilde{X} \xrightarrow{\sim} \tilde{X}_\circ := \text{proj}^{-1}(b_\circ)$, the fiber above the basepoint. Then we have a surjective quotient map $\text{fib}^* : H_T^*(\tilde{X}) \rightarrow H^*(\tilde{X})$ with kernel $(\text{proj}^* H_T^*(\text{pt})_+)$, the ideal generated by the pullback classes of positive degree. Writing this in terms of the polynomial algebra S , we get the Borel picture of the cohomology:

$$\begin{aligned} H_T^*(\tilde{X}) &\twoheadrightarrow H^*(\tilde{X}) \\ S \otimes_{S^W} S &\twoheadrightarrow S / (S_+^W). \end{aligned}$$

Furthermore, consider the projective space $X = K/K^\varpi$. The projection $\pi : \tilde{X} \mapsto X$ induces an inclusion of the equivariant cohomology:

$$\begin{aligned} \pi^* : H_T^*(X) &\hookrightarrow H_T^*(\tilde{X}) \\ S^\varpi \otimes_{S^W} S &\subset S \otimes_{S^W} S, \end{aligned}$$

where S^ϖ denotes the invariants under W^ϖ , the Weyl group of K^ϖ . Similarly for the ordinary cohomology groups.

1.10. Cohomology generators and relations. We introduce coordinates for our cohomology rings. Recall that $BT \rightarrow (\mathbb{P}^\infty)^n$ possesses n dual tautological bundles $\mathcal{O}(0, \dots, 1, \dots, 0)$. We let x_i (resp. y'_i) denote the Chern class of the i^{th} bundle on the first factor (resp. the second factor) of $BT \times_{BK} BT$, and we recall our convention that $y_i := y'_{n+1-i}$. We now get:

$$H_T^*(\tilde{X}) \cong S \otimes_{S^W} S \cong \frac{\mathbb{Q}[x, y]}{\mathfrak{J}} := \frac{\mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n]}{\left(\begin{array}{l} x_1 + \dots + x_n = 0 \\ h(x) = h(y) \quad \forall h \in S^W \end{array} \right)},$$

and:

$$H^*(\tilde{X}) \cong S/(S_+^W) \cong \frac{\mathbb{Q}[x]}{\mathfrak{J}_0} := \frac{\mathbb{Q}[x_1, \dots, x_n]}{(h(x) = 0 \quad \forall h \in \mathbb{Q}[x]_+^W)}.$$

We also have:

$$H_T^*(X) \cong S^\varpi \otimes_{S^W} S \cong \mathbb{Q}[\tilde{x}, y]/\mathfrak{J}^\varpi,$$

where $\tilde{x} := x_1$ and $\mathfrak{J}^\varpi := \mathfrak{J} \cap S^\varpi$, and similarly $H^*(X) := \mathbb{Q}[\tilde{x}]/\mathfrak{J}_0^\varpi$. The restriction maps fib^* are realized by $y = 0$, and π^* is the obvious inclusion.

We give Grobner bases for our ideals. (This is one of the few facts whose generalization to arbitrary K is not known.) Consider the pure lexicographic term-order:

$$y_1 < y_2 < \dots < y_n < x_1 < x_2 < \dots < x_n.$$

Let $h_d(x_{[i]})$ denote the complete symmetric polynomial of degree d in the variables x_1, \dots, x_i , and let $e_d(y)$ denote the elementary symmetric polynomial of degree d in the variables y_1, \dots, y_n . Then the unique reduced Grobner bases of our ideals are:

$$\mathfrak{J} = \left(h_1(x_{[n]}) , e_1(y) , \sum_{i=0}^d (-1)^i h_{d-i}(x_{[n+1-d]}) e_i(y) , d = 1, \dots, n \right) ,$$

$$\mathfrak{J}_0 = \left(h_1(x_{[n]}) , h_2(x_{[n-1]}) , \dots , h_n(x_{[1]}) \right) ,$$

$$\mathfrak{J}^\varpi = \left(e_1(y) , \tilde{x}^n - \tilde{x}^{n-1}e_1(y) + \dots + (-1)^{n-1}\tilde{x}e_{n-1}(y) + (-1)^ne_n(y) \right) ,$$

and $\mathfrak{J}_0^\varpi = (\tilde{x}^n)$, where $\tilde{x} = x_1$.

1.11. Poincare duality. It is possible (e.g., by the work of M.E. Kazarian) to define the Poincare duality isomorphism $H_T^{2\ell}(\tilde{X}) \xrightarrow{\sim} H_{\infty-2\ell}^T(\tilde{X})$, where the right side can be interpreted as the cycles of codimension 2ℓ in the tamely infinite-dimensional space \tilde{X}_T . Any T -invariant subvariety $Y \subset \tilde{X}$ defines a variety $Y_T \subset \tilde{X}_T$, and conversely any finite-codimension subvariety $Y' \subset \tilde{X}_T$ defines a T -invariant subvariety:

$$Y'_\perp = \perp Y' := \text{fib}^{-1}(Y' \cap \tilde{X}_\circ) \subset \tilde{X}.$$

We have $Y = (Y_T)_\perp$. Also $\text{codim}(Y \subset \tilde{X}) = \text{codim}(Y_T \subset \tilde{X}_T)$, as well as $\text{codim}(Y' \subset \tilde{X}_T) = \text{codim}(Y'_\perp \subset \tilde{X})$ provided the intersection $Y' \cap \tilde{X}_\circ$ is transversal.

The geometric operation $Y' \mapsto Y'_\perp$ induces via Poincare duality the cohomology map $\text{fib}^* : H_T^*(\tilde{X}) \rightarrow H^*(\tilde{X})$, where $\text{fib} : \tilde{X} \rightarrow \tilde{X}_T$ is the above fiber map: that is, $\text{fib}^*[Y'] = [Y'_\perp]$. For a polynomial cohomology class $f(x, y) \in \mathbb{Q}[x, y]/\mathfrak{J} \cong H_T^*(\tilde{X})$ we have:

$$\text{fib}^* f = \perp f(x, y) := f(x, 0) \in \mathbb{Q}[x]/\mathfrak{J}_0 \cong H^*(\tilde{X}).$$

The geometric operation $Y \mapsto Y_T$ does not induce a well-defined map $H^*(\tilde{X}) \rightarrow H_T^*(\tilde{X})$, since the equivariant class $[Y_T]$ depends not only upon the homology class $[Y]$, but upon its T -action.

1.12. Demazure operations. For a root $\alpha \in \mathfrak{t}^*$, there are right and left Demazure operations $\partial_\alpha^R, \partial_\alpha^L$ defined on the equivariant cohomology of the flag variety $H_T^*(\tilde{X})$.

We first define the more elementary right operation ∂_α^R acting on the ordinary cohomology $H^*(\tilde{X})$. We have the \mathbb{P}^1 -bundle $\pi_\alpha : K/T \mapsto K/K_\alpha$, inducing on the cohomology $H^*(K/T)$ the pullback map $(\pi_\alpha)^*$ and the push-forward map $(\pi_\alpha)_*$.⁷ Then we let: $\partial_\alpha^R := (\pi_\alpha)^*(\pi_\alpha)_* : H^{2\ell}(\tilde{X}) \rightarrow H^{2\ell-2}(\tilde{X})$.

We can interpret ∂_α^R geometrically by applying Poincare duality to identify $H^{2\ell}(\tilde{X}) \cong H_{2L-2\ell}(\tilde{X})$, where $2L = \dim_{\mathbb{R}}(\tilde{X})$. Namely, given a subvariety $Y \subset K/T = \tilde{X}$, we lift it to $YT \subset K$, right multiply by $K_\alpha \supset T$, and push back down to get the variety $D_\alpha^R(Y) := YK_\alpha/T \subset \tilde{X}$. Then we can express the homology operation in terms of fundamental classes as:

$$\begin{aligned} \partial_\alpha^R : H_{2L-2\ell}(\tilde{X}) &\longrightarrow H_{2L-2\ell+2}(\tilde{X}) \\ [Y] &\longmapsto \begin{cases} [D_\alpha^R Y] & \text{if } \dim_{\mathbb{R}}(D_\alpha^R Y) = \dim_{\mathbb{R}}(Y) + 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To define ∂_α^R on $H_T^*(\tilde{X}) = H^*(\tilde{X}_T)$, we can repeat the above word for word, replacing $\tilde{X} = K/T$ by $\tilde{X}_T = K_T/T$. The equivariant and ordinary Demazure operations, both denoted ∂_α^R , commute with the fiber map $\perp : H_T^*(\tilde{X}) \rightarrow H^*(\tilde{X})$: that is, $\perp \circ \partial_\alpha^R = \partial_\alpha^R \circ \perp$.

⁷The so-called Gysin map, realized on DeRham cohomology as integrating over fibers.

In terms of the coordinates $H_T^*(\tilde{X}) \cong \mathbb{Q}[x, y]/\mathfrak{J}$, the operator ∂_α^R acts as:

$$\partial_\alpha^R f(x, y) := \frac{f(x, y) - f(r_\alpha x, y)}{x_i - x_j},$$

where $\alpha = x_i - x_j$ and $r_\alpha = (ij)$ switches x_i and x_j . Applying \perp to both sides (setting $y = 0$) gives the map on $H^*(\tilde{X}) \cong \mathbb{Q}[x]/\mathfrak{J}_0$.

In case $\alpha = \alpha_i$ is a simple root, we write $\partial_{(i)}^R$. The operators $\partial_{(i)}^R$ satisfy the braid relations $\partial_{(i)}^R \partial_{(i+1)}^R \partial_{(i)}^R = \partial_{(i+1)}^R \partial_{(i)}^R \partial_{(i+1)}^R$, etc., so we can define $\partial_w^R := \partial_{(i_1)}^R \cdots \partial_{(i_\ell)}^R$ for a reduced factorization $w = s_{i_1} \circ \cdots \circ s_{i_\ell} \in W$. Warning: unless α is a simple root, $\partial_\alpha^R \neq \partial_{r_\alpha}^R$.

The left operation ∂_α^L acts only on the equivariant cohomology $H_T^*(\tilde{X})$. To define it, first consider the contraction maps $\kappa_\alpha : K_\alpha \times^T \tilde{X} \rightarrow \tilde{X}$ and $\kappa_\alpha^T : (K_\alpha \times^T \tilde{X})_T \rightarrow \tilde{X}_T$, where $Y_T := ET \times^T Y$ for any left T -space Y . Now let $\partial_\alpha^L := (\kappa_\alpha^T)_* (\kappa_\alpha^T)^*$. For a geometric interpretation, consider a T -invariant subvariety $Y \subset \tilde{X}$, and left multiply by K_α to get the larger T -invariant subvariety $D_\alpha^L(Y) := K_\alpha Y \subset \tilde{X}$. Then we can express the Demazure operation in terms of fundamental classes as:

$$\begin{aligned} \partial_\alpha^L : H_{\infty-2\ell}^T(\tilde{X}) &\longrightarrow H_{\infty-2\ell+2}^T(\tilde{X}) \\ [Y]_T &\longmapsto \begin{cases} [D_\alpha^L Y]_T & \text{if } \dim_{\mathbb{R}}(D_\alpha^L Y) = \dim_{\mathbb{R}}(Y) + 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the polynomial ring $\mathbb{Q}[x, y]$, we have:

$$\partial_\alpha^L f(x, y) := (-1) \frac{f(x, y) - f(x, r_\alpha y)}{y_i - y_j}.$$

where $\alpha = y_i - y_j$ and $r_\alpha = (ij)$ switches y_i and y_j . (N.B. the minus in front!) We once again have ∂_w^L , and in general $\partial_\alpha^L \neq \partial_{r_\alpha}^L$.

Let us emphasize that in all the above, α can be any root, not necessarily simple. We will sometimes write ∂_α^x , ∂_α^y instead of ∂_α^R , ∂_α^L , respectively.

Finally, we consider the Gysin push-forward of the projection $\pi : \tilde{X} \rightarrow X$:

$$\begin{aligned} \pi_* : H_T^{2\ell}(\tilde{X}) &\twoheadrightarrow H_T^{2\ell-2M}(X) \\ f(x, y) &\longmapsto \partial_{(\varpi)}^x f(x, y), \end{aligned}$$

where $2M = \dim_{\mathbb{R}}(\tilde{X}) - \dim_{\mathbb{R}}(X)$, and we denote $\partial_{(\varpi)}^x := \partial_w^R$, the right Demazure operator of $w = w^\varpi$, the longest element of the maximal parabolic $W^\varpi = \text{Stab}_W(\varpi) = S_1 \times S_{n-1}$. In fact, given $f(x, y) \in H_T^*(\tilde{X})$, we may take its Grobner normal form $\bar{f}(x, y)$ and expand it as:

$$\bar{f}(x, y) = \sum_{a=(a_2, \dots, a_n)} \bar{f}_a(\tilde{x}, y) x_2^{a_2} \cdots x_n^{a_n},$$

where $\tilde{x} = x_1$. Then for $a = (n-2, \dots, 2, 1, 0)$ we have:

$$\partial_{(\varpi)}^x f(x, y) = \partial_{(\varpi)}^x \bar{f}(x, y) = \bar{f}_a(\tilde{x}, y) \mod \mathfrak{J}^\varpi.$$

1.13. Schubert polynomials. The basepoint $\text{pt} := \text{id } T \in K/T = \tilde{X}$ corresponds to $\text{pt}_T \subset \tilde{X}_T$, which is isomorphic to the diagonal $\text{diag}(BT) \subset BT \times_{BK} BT$.

The equivariant class of the basepoint, i.e., the class of the diagonal, corresponds via Poincare duality to the double Schubert polynomial:

$$\mathfrak{S}_{w_0}(x, y) := \prod_{i+j \leq n} (x_i - y_j) = [\text{pt}]_T \in H_T^{2L}(\tilde{X}),$$

where $2L := \dim_{\mathbb{R}}(\tilde{X}) = 2\binom{n}{2}$.

Proof. Recall that $BT \times_{BK} BT$ maps to the space:

$$B(S^1)^n \times_{BU_n} B(S^1)^n = \{[v_1, \dots, v_n] \times [v'_1, \dots, v'_n]\},$$

where $v_i, v'_i \in \mathbb{P}^\infty$ and $[v_1, \dots, v_n], [v'_1, \dots, v'_n]$ run over orthogonal 1-dimensional splittings of some n -dimensional space inside the Hermitean space \mathbb{C}^∞ . The natural coordinates for $H_T^*(\tilde{X}) = H^*(BT \times_{BK} BT)$ are the Chern classes x_1, \dots, x_n and y'_1, \dots, y'_n , which can be defined by the loci: $x_i = [v_i \in H]$ and $y'_i = [v'_i \in H]$, where $H \subset \mathbb{C}^\infty$ is a hyperplane.

For $i, j \in [1, n]$, consider the locus on which v_i is orthogonal to v'_j , the divisor:

$$D_{ij} := \{v_i \cdot v'_j = 0\}.$$

The diagonal $\Delta BT \subset BT \times_{BK} BT$ is the locus:

$$\Delta BT = \{v_i = v'_i \mid i = 1, \dots, n\} = \bigcap_{i < j} D_{ij},$$

a transversal intersection. Demazure shows that we can express our divisors as: $[D_{ij}] = x_i - y'_j$, so:

$$[\text{pt}]_T = [\Delta BT] = \prod_{i < j} (x_i - y'_j).$$

Substituting $y_i := y'_{n+1-i}$ gives the desired formula. \square

For general $w \in W$, the equivariant classes corresponding to the Schubert varieties $\tilde{X}_w := \text{closure}(B \cdot wT) \subset \tilde{X}$ are given by the double Schubert polynomials $\mathfrak{S}_w(x, y)$. These can be computed from $\mathfrak{S}_{w_0}(x, y)$ via the right recurrence:

$$\mathfrak{S}_{ws_i}(x, y) := \partial_{(i)}^R \mathfrak{S}_w(x, y) \quad \text{if } \ell(ws_i) = \ell(w) - 1;$$

or alternatively the left recurrence:

$$\mathfrak{S}_{s_i w}(x, y) := \partial_{(i)}^L \mathfrak{S}_w(x, y) \quad \text{if } \ell(s_i w) = \ell(w) - 1.$$

Our constructions imply the non-trivial fact that these two recurrences produce the same polynomial \mathfrak{S}_w for each w . This fact is equivalent to the identity:

$$\mathfrak{S}_w(x, y) = (-1)^{\ell(w)} \mathfrak{S}_{w^{-1}}(y, x).$$

Setting $y = 0$ gives us the single Schubert polynomials $\mathfrak{S}_w(x) := \mathfrak{S}_w(x, 0)$, representing the Schubert classes in the ordinary cohomology $H^*(\tilde{X})$. These polynomials can be computed directly from $\mathfrak{S}_{w_0}(x) = x_1^{n-1}x_2^{n-2} \cdots x_{n-2}^2x_{n-1}$ via the right recurrence: however, the left recurrence no longer makes sense.

Example: For $G = \mathrm{SL}_3\mathbb{C}$, we have the equivariant cohomology ring of the flag variety $\tilde{X} = \mathrm{Flag}(\mathbb{C}^3)$:

$$H_T^*(\tilde{X}) \cong \frac{\mathbb{Q}[x, y]}{\mathfrak{J}} \cong \frac{\mathbb{Q}[x_1, x_2, x_3, y_1, y_2, y_3]}{\left(\begin{array}{l} x_1+x_2+x_3, \quad y_1+y_2+y_3 \\ (x_1^2+x_1x_2+x_2^2) - (x_1+x_2)(y_1+y_2+y_3) + (y_1y_2+y_1y_3+y_2y_3) \\ x_1^3 - x_1^2(y_1+y_2+y_3) + x_1(y_1y_2+y_1y_3+y_2y_3) - y_1y_2y_3 \end{array} \right)},$$

and the double Schubert polynomials are:

$$\begin{aligned} \mathfrak{S}_{w_0} &= (x_1-y_1)(x_1-y_2)(x_2-y_2) \\ \mathfrak{S}_{s_1s_2} &= (x_1-y_1)(x_1-y_2), \quad \mathfrak{S}_{s_2s_1} = (x_1-y_1)(x_2-y_1) \\ \mathfrak{S}_{s_1} &= x_1-y_1, \quad \mathfrak{S}_{s_2} = x_1+x_2-y_1-y_2, \quad \mathfrak{S}_1 = 1. \end{aligned}$$

Setting $y_1 = y_2 = y_3 = 0$ gives the ordinary cohomology ring:

$$H^*(\tilde{X}) \cong \frac{\mathbb{Q}[x]}{\mathfrak{J}_0} \cong \frac{\mathbb{Q}[x_1, x_2, x_3]}{(x_1+x_2+x_3, x_1^2+x_1x_2+x_2^2, x_1^3)},$$

with the basis of single Schubert polynomials:

$$\mathfrak{S}_{w_0} = x_1^2x_2, \quad \mathfrak{S}_{s_1s_2} = x_1^2, \quad \mathfrak{S}_{s_2s_1} = x_1x_2, \quad \mathfrak{S}_{s_1} = x_1, \quad \mathfrak{S}_{s_2} = x_1+x_2, \quad \mathfrak{S}_1 = 1.$$

For the projective space $X = \mathbb{P}^2$, we take $\tilde{x} := x_1$, and we have:

$$H_T^*(X) \cong \frac{\mathbb{Q}[\tilde{x}, y]}{\mathfrak{J}^\varpi} \cong \frac{\mathbb{Q}[\tilde{x}, y_1, y_2, y_3]}{\left(\begin{array}{l} y_1+y_2+y_3 \\ \tilde{x}^3 - \tilde{x}^2(y_1+y_2+y_3) + \tilde{x}(y_1y_2+y_1y_3+y_2y_3) - y_1y_2y_3 \end{array} \right)}$$

and $H^*(X) \cong \mathbb{Q}[\tilde{x}]/\mathfrak{J}_0^\varpi \cong \mathbb{Q}[\tilde{x}]/(\tilde{x}^3)$, with the Schubert polynomials:

$$\mathfrak{S}_{s_2s_1} = (\tilde{x}-y_1)(\tilde{x}-y_2), \quad \mathfrak{S}_{s_1} = \tilde{x}-y_1, \quad \mathfrak{S}_1 = 1.$$

1.14. Weyl group actions on cohomology. The non-holomorphic right action of W on $\tilde{X} = K/T$, namely $kT \cdot w := k\overset{\circ}{w}T$, induces a right action on $H_T^*(\tilde{X})$. Recall that each reflection is an orientation-reversing map.

The naive notion of a left W -action, $w \cdot kT := \overset{\circ}{w}kT$, is not well-defined. However, $w \cdot Y := \overset{\circ}{w}Y$ does define an operation on T -invariant subvarieties $Y \subset \tilde{X}$, so we get a left W -action on $H_T^*(\tilde{X})$.

Considering \tilde{X}_T as $BT \times_{BK} BT$, we can see these two actions as the natural W -actions on the two factors. That is, we have the natural action of W on \mathfrak{t}^* , written $\lambda \mapsto w(\lambda)$ or in coordinates $w(x_i) = x_{w(i)}$, and this induces

the usual W -action on $S = \text{Sym}(\mathfrak{t}^*)$. If a cohomology class is Poincare dual to a subvariety $Y \subset \tilde{X}$, so that:

$$[Y]_T = f(x, y) = f(x, w_0(y')) \in S \otimes_{S^W} S,$$

where y' are the natural coordinates and $y = w_0(y')$, $y_i = y'_{n+1-i}$ according to our convention, then:

$$[w \cdot Y]_T = f(x, w_0(wy')) = f(x, w^*(y)) \quad , \quad [Y \cdot w]_T = (-1)^{\ell(w)} f(w^{-1}(x), y),$$

where $w^* := w_0 w w_0$ is the Weyl duality, and the sign adjusts orientation.

Now consider the equivariant class of a T -fixed point $w = wT \in \tilde{X}$. If $\text{pt} = \text{id}T$ is the basepoint with $[\text{pt}]_T = \mathfrak{S}_{w_0} = \prod_{i+j \leq n} (x_i - y_j)$, then $w = w \cdot \text{pt} = (-1)^{\ell(w)} \text{pt} \cdot w$, so we have:

$$[w]_T = \mathfrak{S}_{w_0}(x, w^*(y)) = (-1)^{\ell(w)} \mathfrak{S}_{w_0}(w^{-1}(x), y) \mod \mathfrak{J}.$$

The two polynomials on the right are not equal, but equivalent modulo the ideal.

Considering $H_T^*(X) = S^\varpi \otimes_{S^W} S \cong \mathbb{Q}[\tilde{x}, y]/\mathfrak{J}^\varpi$, we have T -fixed points $u_i = u_i K^\varpi \in X = K/K^\varpi$, where $u_i = s_i s_{i-1} \cdots s_2 s_1$ for $i = 0, \dots, n-1$ are the minimal coset representatives of W/W^ϖ . Their classes are:

$$[u_i]_T = \mathfrak{S}_{u_{n-1}}(\tilde{x}, u_i^*(y)) = \prod_{j \neq n-i} (\tilde{x} - y_j).$$

Note that $[u_i \in X]_T = \pi_* [u_i \in \tilde{X}]_T$.

2. AFFINE SCHUBERT VARIETIES AND FACTORIZATION

2.1. Minimal parabolic subgroups. The elementary unipotent subgroup associated to a root $\alpha = e_i - e_j$ is $U_\alpha := \{I + x E_{ij} \mid x \in \mathbb{C}\} \cong (\mathbb{C}, +)$, where E_{ij} is an off-diagonal coordinate matrix. A minimal parabolic is the subgroup generated by a negative elementary subgroup and the Borel: $P_\alpha = U_{-\alpha} B$, where α is a positive root. For a simple root $\alpha = \alpha_i = e_i - e_{i+1}$, we write the parabolic as $P_{(i)}$. We also have the compact form $K_\alpha := K \cap P_\alpha$ consisting of matrices which are diagonal except for a copy of U_2 in the ij -block, so that $T \subset K_\alpha$.

We make the corresponding definitions for the loop group. If $\epsilon_0 : G[[t]] \rightarrow G$, $f(t) \mapsto f(0)$ is the evaluation map, we define the Iwahori subgroup $\hat{B} := \epsilon_0^{-1}(B)$, which is also a Borel subgroup of $G((t))$ thought of as a Kac-Moody group. For $\alpha = e_i - e_j$ and $k \in \mathbb{Z}$, we have the affine root subgroup $U_{(\alpha, k)} := \{I + x t^k E_{ij} \mid x \in \mathbb{C}\}$. The positive affine roots are of the form $(\alpha, 0)$ and $(\pm\alpha, k)$ for α a positive root of G and $k > 0$. We define the minimal parabolic $\hat{P}_{(\alpha, k)} := U_{(-\alpha, -k)} \hat{B}$ whenever (α, k) is positive. In particular, for the simple roots $\hat{\alpha}_i := (e_i - e_{i+1}, 0)$ for $i = 1, \dots, n-1$, and $\hat{\alpha}_0 := (e_n - e_1, 1)$, we write the minimal parabolic as $\hat{P}_{(i)}$.

The compact form of an affine parabolic is: $K_{(\alpha, k)} := LK \cap \hat{P}_{(\alpha, k)}$. Crucially, $K_{(\alpha, k)}$ is once again a finite-dimensional compact Lie group. Indeed,

$K_{(e_i - e_j, k)}$ for $i < j$ is generated by $T \subset LK$ and a copy of SU_2 in the ij -block, in the form:

$$\begin{bmatrix} a & -\bar{b}t^k \\ bt^{-k} & \bar{a} \end{bmatrix},$$

where $a, b \in \mathbb{C}$ with $a\bar{a} + b\bar{b} = 1$. (This lies in SU_2 for all $t \in S^1$, since $t^{-1} = \bar{t}$.) Also $K_{(e_j - e_i, k)} = K_{(e_i - e_j, -k)}$, the complex conjugate of the above.

We have $K_{(\alpha, 0)} = K_\alpha \subset K \subset LK$ for all the finite roots α , and the groups $K_{(0)}, K_{(1)}, \dots, K_{(n-1)}$ corresponding to the affine simple roots are all isomorphic to each other via the diagram automorphism of $\widehat{\text{Dyn}}$. For example, for $G = SL_3\mathbb{C}$, the simple-root compact parabolics are:

$$K_{(0)} = \begin{bmatrix} a & 0 & -\bar{b}t^{-1} \\ 0 & c & 0 \\ bt & 0 & \bar{a} \end{bmatrix}, \quad K_{(1)} = \begin{bmatrix} a & -\bar{b} & 0 \\ b & \bar{a} & 0 \\ 0 & 0 & c \end{bmatrix}, \quad K_{(2)} = \begin{bmatrix} c & 0 & 0 \\ 0 & a & -\bar{b} \\ 0 & b & \bar{a} \end{bmatrix},$$

where $a, b, c \in \mathbb{C}$ with $(a\bar{a} + b\bar{b})c = 1$.

2.2. Schubert varieties in ΩK . We say the factorization $v = v_1 \cdots v_r \in \widehat{W}$ is reduced if $\ell(v) = \ell(v_1) + \cdots + \ell(v_r)$, and we write this as: $v = v_1 \circ \cdots \circ v_r$.

Let $v = s_{i_1} \circ \cdots \circ s_{i_\ell} \in \widehat{W}$ be a reduced factorization into affine simple reflections. The spaces we define below will be independent of the choice of factorization for v . Define the algebraic group Schubert variety:

$$\widehat{G}_v := \widehat{P}_{(i_1)} \cdots \widehat{P}_{(i_r)} \subset G((t));$$

and the compact group Schubert variety:

$$LK_v := LK \cap \widehat{G}_v = K_{(i_1)} \cdots K_{(i_r)} \subset LK.$$

Also define $\widehat{G}_{\text{id}} := \widehat{B}$ and $LK_{\text{id}} := T$. For $w \in W$, we will write $K_w \subset K$ instead of $LK_w \subset K \subset LK$.

The Schubert variety in the usual sense (inside the affine Grassmannian) is denoted $\widehat{\text{Gr}}_v$ in the algebraic category and ΩK_v in the topological category:

$$\begin{aligned} \widehat{\text{Gr}}_v &:= (\widehat{G}_v \cdot G[[t]]) / G[[t]] \subset G((t)) / G[[t]] \cong \widehat{\text{Gr}} \\ &\cong \Omega K_v := (LK_v \cdot K) / K \subset LK / K \cong \Omega K. \end{aligned}$$

Put another way, let $\nu : LK \rightarrow \Omega K$, $f(t) \mapsto f(t)f(1)^{-1}$ be the base-point normalizing map: then $\Omega K_v = \nu(LK_v)$.

We may also obtain the Schubert varieties in $\widehat{\text{Gr}} \cong \Omega K$ as the \widehat{B} -orbit closures: $\widehat{\text{Gr}}_v = \text{closure}(\widehat{B} \cdot \overset{\circ}{v} \subset \widehat{\text{Gr}})$, where $\overset{\circ}{v} \in G((t))$ is a representative of v . However, this definition has no compact counterpart: the Schubert cell $\widehat{B} \cdot \overset{\circ}{v}$ is non-compact, so it cannot be the orbit of any compact group.

Note that $\Omega K_v = \Omega K_u$ whenever $vW = uW$; thus the distinct Schubert varieties in ΩK are indexed by the right cosets $\widehat{W}/W = \{t^\lambda W\}_{\lambda \in Q^\vee}$. Another useful set of coset representatives is given by the elements of minimal

length $m^\lambda := \min(t^\lambda W)$: we call these the right-minimal elements of \widehat{W} . We denote:

$$\Omega K_\lambda := \Omega K_{t^\lambda} = \Omega K_{m^\lambda} \quad \text{and} \quad LK_\lambda := LK_{m^\lambda}.$$

We have a homeomorphism:⁸

$$\Omega K_\lambda := (LK_\lambda \cdot K)/K \cong LK_\lambda/T.$$

Clearly, LK_v and ΩK_v are finite-dimensional, and one can show that $\widehat{\text{Gr}}_\lambda \cong \Omega K_\lambda$ is a rational complex algebraic variety with $\dim_{\mathbb{R}} = 2\ell(m^\lambda)$. Indeed, one can construct ΩK explicitly as a CW-complex with one even-dimensional cell $\widehat{B} \cdot t^\lambda \cong \mathbb{C}^{\ell(m^\lambda)}$ corresponding to each Schubert variety, so that the fundamental cycles form a \mathbb{Q} -linear basis of the loop-group homology:

$$H_*(\Omega K) = \bigoplus_{\lambda \in Q^\vee} \mathbb{Q}[\Omega K_\lambda].$$

2.3. Schubert varieties in ΩK^{ad} . We will also need the Schubert subvarieties of ΩK^{ad} , indexed by $\widehat{W}/W \cong P^\vee \cong \Sigma \times Q^\vee$, where $\sigma \in \Sigma$ are the elements of length zero. We say $v = v_1 \circ \cdots \circ v_\ell \in \widehat{W}$ is a reduced factorization into extended simple roots if we take $v_j \in \{s_0, s_1, \dots, s_{n-1}\} \cup \Sigma$, and the product is reduced (using the length function on \widehat{W}). For each $\sigma \in \Sigma$ and its representative $\overset{\circ}{\sigma} \in LK$, we let $K_\sigma := \overset{\circ}{\sigma}T = T\overset{\circ}{\sigma} \subset LK$. Now for any $v \in \widehat{W}$, we can define LK_v , ΩK_v , etc., word-for-word as before, except that now $\Omega K_v \subset \Omega K^{\text{ad}}$, etc. For $v \in \widehat{W}$, we consider $\Omega K_v \subset \Omega K \subset \Omega K^{\text{ad}}$.

To any $\lambda \in P^\vee$ we associate $\widehat{\lambda} \in Q^\vee$ as follows. Take $m^\lambda = \sigma_\lambda \widehat{m}^\lambda$ with $\sigma_\lambda \in \Sigma$ and $\widehat{m}^\lambda \in \widehat{W}$. Since \widehat{m}^λ is again right-minimal, we have $\widehat{m}^\lambda = m^{\widehat{\lambda}}$ for some $\widehat{\lambda}$. Further, let $\eta_\lambda = \sigma_\lambda(0)$ be the minuscule fundamental weight (or the zero weight) corresponding to σ_λ . Then we have:

$$\Omega K_\lambda = \Omega K_{m^\lambda} = K_{\sigma_\lambda} \cdot \Omega K_{\widehat{m}^\lambda} = \sigma_\lambda \cdot \Omega K_{\widehat{\lambda}} \subset \Omega K^{\text{ad}}.$$

2.4. Bott-Samelson varieties. Now suppose $v = v_1 v_2 \cdots v_r$ is any factorization of $v \in \widehat{W}$ (or $v \in \widetilde{W}$), not necessarily reduced. We define the Bott-Samelson variety as the quotient:

$$\Omega K_{v_1, \dots, v_r} = LK_{v_1} \overset{T}{\times} \cdots \overset{T}{\times} LK_{v_r} / T := \frac{LK_{v_1} \times \cdots \times LK_{v_r}}{T \times \cdots \times T}$$

where $LK_{v_1} \times \cdots \times LK_{v_r}$ has the following right action of T^r :

$$(x_1, \dots, x_r) \cdot (t_1, \dots, t_r) := (x_1 t_1, t_1^{-1} x_2 t_2, \dots, t_{r-1}^{-1} x_r t_r).$$

This is homeomorphic to the usual algebraic definition of the Bott-Samelson variety (cf. Demazure). Indeed $\Omega K_{v_1, \dots, v_r}$ is a complex algebraic variety with $\dim_{\mathbb{R}} = 2\ell(v_1) + \cdots + 2\ell(v_r)$.

⁸Indeed, the obvious map $LK_\lambda/T \rightarrow (LK_\lambda \cdot K)/K$ is a continuous bijection between compact Hausdorff spaces, and so is a homeomorphism.

Now let $v = v_1 \circ \cdots \circ v_r$ with v right-minimal (which implies that v_r is also right-minimal). Then we have $\Omega K_v \cong LK_v/T$, and we can define the contraction map:

$$\begin{aligned} \text{cont} : \quad \Omega K_{v_1, \dots, v_r} &\rightarrow \Omega K_v \\ (x_1, \dots, x_r) &\mapsto x_1 \cdots x_r. \end{aligned}$$

This map is continuous and surjective, and it is birational, meaning that $\text{cont}^{-1}(U) \cong U$ for an open neighborhood U of almost any point of ΩK_v . Therefore the induced homology map $\text{cont}_* : H_*(\Omega K_{v_1, \dots, v_r}) \rightarrow H_*(\Omega K)$ takes the fundamental class of the Bott-Samelson variety to that of the Schubert variety: $\text{cont}_*[\Omega K_{v_1, \dots, v_r}] = [\Omega K_v]$.

2.5. Factorization. Recall that $m^\lambda := \min(t^\lambda W)$; that $\{m^\lambda\}_{\lambda \in Q^\vee}$, where Q^\vee is the coroot lattice, are coset representatives for \widehat{W}/W ; and that the Schubert varieties of ΩK are $\Omega K_\lambda := \Omega K_{m^\lambda} = \Omega K_{t^\lambda}$ for $\lambda \in Q^\vee$.

Furthermore, $\{\widehat{m}^\lambda\}_{\lambda \in P^\vee}$, where P^\vee is the coweight lattice, are coset representatives for \widehat{W}/W , where \widehat{W} is the extended affine Weyl group. For $\lambda \in P^\vee$, we decompose $m^\lambda = \sigma_\lambda \widehat{m}^\lambda$ with $\ell(\sigma_\lambda) = 0$ and $\widehat{m}^\lambda \in \widehat{W}$, and we define $\widehat{\lambda} \in Q^\vee$ by $m^{\widehat{\lambda}} = \min(\widehat{m}^\lambda W)$.

THEOREM A: *Suppose we have coweights $\lambda, \mu, \nu \in P^\vee$ with $\mu \in P_-^\vee$ anti-dominant, such that $m^\lambda t^\mu = m^\nu \in \widehat{W}$ and $\ell(m^\lambda) + \ell(t^\mu) = \ell(m^\nu)$. Then the Schubert homology classes in the Pontryagin ring $H_*(\Omega K)$ satisfy:*

$$[\Omega K_{\widehat{\lambda}}] \cdot [\Omega K_{\widehat{\mu}}] = [\Omega K_{\widehat{\nu}}].$$

Proof. For any $\mu \in P_-^\vee$ (meaning $-\mu \in P_+^\vee$ is dominant), we have $m^\mu = t^\mu$ and $\ell(wt^\mu W) \leq \ell(t^\mu W)$ for all $w \in W$. Thus $K \cdot LK_\mu \cdot K = LK_\mu \cdot K$, meaning that $LK_\mu \cdot K$ has a left K -action.

Given $m^\lambda \circ t^\mu = m^\nu$ as in the Theorem, we can write the corresponding Bott-Samelson variety as:

$$\Omega K_{\lambda, \mu} := LK_\lambda \overset{T}{\times} LK_\mu / T \cong (LK_\lambda \cdot K) \overset{K}{\times} (LK_\mu \cdot K) / K.$$

Now we can define a homeomorphism by the two inverse maps:

$$\begin{array}{ccc} \Omega K_{\lambda, \mu} & \xleftrightarrow{\sim} & \Omega K_\lambda \times \Omega K_\mu \\ \parallel & & \parallel \\ (LK_\lambda \cdot K) \overset{K}{\times} (LK_\mu \cdot K) / K & & (LK_\lambda \cdot K) / K \times (LK_\mu \cdot K) / K \\ (f_1(t), f_2(t)) & \longmapsto & (f_1(t) f_1(1)^{-1}, f_1(1) f_2(t)) \\ (f_1(t), f_2(t)) & \longleftarrow & (f_1(t), f_2(t)). \end{array}$$

Furthermore, we have the commutative triangle:

$$\begin{array}{ccc} \Omega K_{\lambda,\mu} & \xrightarrow{\sim} & \Omega K_{\lambda} \times \Omega K_{\mu} \\ \text{cont} \searrow & & \downarrow \text{mult} \\ & & \Omega K_{\nu} \end{array},$$

where mult is loop group multiplication and cont is the birational contraction map. Taking the induced maps on the fundamental classes in the homology $H_*(\Omega K)$ gives:

$$[\Omega K_{\lambda}] \cdot [\Omega K_{\mu}] = \text{mult}_*[\Omega K_{\lambda} \times \Omega K_{\mu}] = \text{cont}_*[\Omega K_{\lambda,\mu}] = [\Omega K_{\nu}].$$

Now we have $[\Omega K_{\lambda}] = [\sigma_{\lambda}] \cdot [\Omega K_{\hat{\lambda}}]$, where $[\sigma_{\lambda}]$ is a one-point homology class, and:

$$[\sigma_{\lambda}] \cdot [\Omega K_{\hat{\lambda}}] \cdot [\sigma_{\mu}] \cdot [\Omega K_{\hat{\mu}}] = [\sigma_{\nu}] \cdot [\Omega K_{\hat{\nu}}].$$

But Pontryagin multiplication is commutative and $\sigma_{\lambda}\sigma_{\mu} = \sigma_{\nu}$, so we can cancel the one-point homology classes from the above equation. \square

We make some comments on the proof. The key to defining the homeomorphism $\Omega K_{\lambda,\mu} \xrightarrow{\sim} \Omega K_{\lambda} \times \Omega K_{\mu}$ is to pick out the K -factor of a loop $f_1(t) \in LK_{\lambda} \cdot K$: it is the basepoint $f_1(1)$. This allows us to disentangle the K -twisted product. Let us see why this proof cannot be adapted to the algebraic category (and so does not prove the analogous results about Demazure modules). Algebraically, we can again write the Bott-Samelson variety as:

$$\widehat{\text{Gr}}_{\lambda,\mu} := (\widehat{G}_{\lambda} \cdot G[[t]]) \overset{G[[t]]}{\times} (\widehat{G}_{\mu} \cdot G[[t]]) / G[[t]],$$

which is homeomorphic to $\Omega K_{\lambda,\mu}$. But now there is no algebraic way to pick out the $G[[t]]$ -factor of $f_1(t) \in \widehat{G}_{\lambda} \cdot G[[t]]$, so we cannot disentangle the $G[[t]]$ -twisted product. In fact, $\widehat{\text{Gr}}_{\lambda,\mu}$ and $\widehat{\text{Gr}}_{\lambda} \times \widehat{\text{Gr}}_{\mu}$ have non-isomorphic complex structures which are connected by deforming the complex structure on a fixed underlying topological space, as in the recent work of Beilinson-Drinfeld (see [G]).

3. AFFINE SCHUBERT POLYNOMIALS

3.1. Homology symmetric algebra. Let $X = K/K^{\varpi} \cong \mathbb{P}^{n-1}$, where $\varpi := \varpi_1^{\vee}$, the first fundamental coweight. We have the singular homology:

$$H_*(X) := H_*(\mathbb{P}^{n-1}, \mathbb{Q}) = \mathbb{Q}1 \oplus \mathbb{Q}h_1 \oplus \cdots \oplus \mathbb{Q}h_{n-1},$$

where 1 is the 0-dimensional homology class of a point and h_i is the $2i$ -dimensional homology class of a projective i -plane $\mathbb{P}^i \subset \mathbb{P}^{n-1}$. For our purposes, $H_*(X)$ is a vector space with no natural multiplication.

Recall that $H_*(X^m) \cong H_*(X)^{\otimes m}$. Consider the embedding $\text{pt} : X^m \xrightarrow{\sim} \text{pt} \times X^m \subset X^{m+1}$: by abuse of notation, pt means either the basepoint of X or the inclusion map. Define the symmetric power $S^m H_*(X)$ as the quotient of $H_*(X)^{\otimes m}$ by the commutativity relations, and define the symmetric

algebra as the direct limit

$$SH_*(X) = \varinjlim S^m H_*(X)$$

under the above embeddings, which induce:

$$\text{pt}^* : S^m H_*(X) \hookrightarrow S^{m+1} H_*(X) \quad , \quad y_1 \cdot y_2 \cdots y_m \mapsto 1 \cdot y_1 \cdot y_2 \cdots y_m .$$

In particular, the unit is $1 = 1 \cdot 1 = \cdots$ and any $y \in H_*(X)$ is identified with $y = 1 \cdot y = 1 \cdot 1 \cdot y = \cdots$. In our case, we clearly have $SH_*(X) \cong \mathbb{Q}[h_1, \dots, h_{n-1}]$, a polynomial ring.

3.2. Bott's isomorphism Ψ . Bott defines⁹ the mapping:

$$\begin{aligned} \Psi : K &\rightarrow \Omega K \\ k &\mapsto k t^\varpi k^{-1} t^{-\varpi} . \end{aligned}$$

This descends to a map $\Psi : X = K/K^\varpi \rightarrow \Omega K$, and it induces a \mathbb{Q} -linear map $\Psi_* : H_*(X) \rightarrow H_*(\Omega K)$ and a ring homomorphism $\Psi_{**} : SH_*(X) \rightarrow H_*(\Omega K)$.

THEOREM: *The Bott map Ψ_{**} is an isomorphism from the polynomial ring $\mathbb{Q}[h_1, \dots, h_{n-1}] = SH_*(\mathbb{P}^{n-1})$ to the Pontryagin ring $H_*(\Omega K)$. This isomorphism preserves the dimension grading, where $\dim(h_i) = 2i$.*

In the sections that follow, we will paraphrase Bott's proof of this theorem (at least of the surjectivity part). Our aim is to explicitly describe for each $\lambda \in Q^\vee$ the unique polynomial $\widehat{\mathfrak{S}}_\lambda \in SH_*(X)$ with $\Psi_{**}(\widehat{\mathfrak{S}}_\lambda) = [\Omega K_\lambda]$.

Note: For G of other types, the map Ψ is defined in terms of a weight ϖ with the property that for each simple root $\alpha_1, \alpha_2, \dots$ of G , there exists some $w \in W$ such that $\langle \alpha_i, w(\varpi) \rangle = 1$. (There is always some fundamental coweight $\varpi = \varpi_i^\vee$ with this property.) Then we take $X = K/K^\varpi$ with $H_*(X) \cong \mathbb{Q}1 \oplus \mathbb{Q}h_1 \oplus \cdots \oplus \mathbb{Q}h_r$, where the basis elements h_j correspond to the cosets in the parabolic Bruhat order W/W^ϖ .

The map Ψ_{**} is always surjective, as will be clear from our arguments below. However, Ψ_{**} might not be injective, so that $\widehat{\mathfrak{S}}_\lambda = \Psi_{**}^{-1}[\Omega K_\lambda]$ is no longer a unique polynomial. Rather, $H^*(\Omega K)$ will be isomorphic to a quotient of the polynomial ring $SH_*(X) = \mathbb{Q}[h_1, \dots, h_r]$ by a certain polynomial ideal \mathfrak{I} , and $\widehat{\mathfrak{S}}_\lambda$ will be well-defined modulo \mathfrak{I} . The algebraic variety $\text{Spec } H_*(\Omega K) \subset \mathbb{C}^r$ defined by the ideal \mathfrak{I} is described in Peterson's notes, and is often called the Peterson variety.

3.3. Bott's map Φ . We will find it convenient to rephrase Bott's isomorphism in terms of the Z -shifted map:

$$\begin{aligned} \Phi : K/K^\varpi &\rightarrow \Omega K^{\text{ad}} \\ k &\mapsto k t^\varpi k^{-1} . \end{aligned}$$

⁹Our mappings Ψ, Φ are in Bott's notation g_s, f_s respectively.

We have induced maps $\Phi_* : H_*(X) \rightarrow H_*(\Omega K^{\text{ad}})$ and $\Phi_*^m : H_*(X^m) \rightarrow H_*(\Omega K^{\text{ad}})$. However, since $\Phi_*(1) = t^\varpi \neq 1 \in \Omega K^{\text{ad}}$, the maps Φ_*^m do not intertwine with the inclusions $S^m H_*(X) \hookrightarrow S^{m+1} H_*(X)$, and do *not* induce a map $\Phi_{**} : SH_*(X) \rightarrow H_*(\Omega K^{\text{ad}})$.

Instead, we consider the normalized maps

$$\widehat{\Phi}^m(x_1, \dots, x_m) := \Phi(x_1) \cdots \Phi(x_m) \cdot t^{-m\varpi},$$

which do induce a map $\widehat{\Phi}_{**} : SH_*(X) \rightarrow H_*(\Omega K)$. Because of the commutativity of the Pontryagin product, we clearly have $\widehat{\Phi}_{**} = \Psi_{**}$.

LEMMA: *Let $\lambda \in P^\vee$ correspond to $\widehat{\lambda} \in Q^\vee$, and let $[Y] \in H_*(X^m)$. Then:*

$$\Phi_*^m[Y] = [\Omega K_\lambda] \in H_*(\Omega K^{\text{ad}}) \implies \Psi_{**}[Y] = [\Omega K_{\widehat{\lambda}}] \in H_*(\Omega K).$$

Proof: Let $[t^\lambda]$ denote the zero-dimensional homology class of the point $t^\lambda \in \Omega K^{\text{ad}}$. Let η be the minimal dominant weight with $\eta = m\varpi \bmod Q^\vee$: since t^η and $t^{m\varpi}$ lie in the same component of ΩK^{ad} , we have $[t^\eta] = [t^{m\varpi}]$. Indeed, $\text{Im}(\Phi^m)$ lies completely in the $(m\varpi)$ -component of ΩK^{ad} , and $\Phi_*^m[Y] = [\Omega K_\lambda]$, so we have $\lambda = m\varpi \bmod Q^\vee$ also, and $\Omega K_\lambda = \Omega K_{\widehat{\lambda}} \cdot t^\eta$. Then:

$$\Phi_*^m[Y] = [\Omega K_\lambda] = [\Omega K_{\widehat{\lambda}}] \cdot [t^\eta] = [\Omega K_{\widehat{\lambda}}] \cdot [t^{m\varpi}].$$

Hence $\Psi_{**}[Y] = \widehat{\Phi}_{**}[Y] = \widehat{\Phi}_*^m[Y] = [\Omega K_\lambda] \cdot [t^{-m\varpi}] = [\Omega K_{\widehat{\lambda}}]$, proving the Lemma.

3.4. Schubert varieties pulled back to X^m . Given $v = m^\lambda \in \widehat{W}$ and a reduced factorization into simple reflections, we can group the factorization into elements of W separated by the affine simple reflection s_0 :

$$v = w_1 \circ s_0 \circ w_2 \circ s_0 \circ \cdots \circ w_r \circ s_0,$$

where $w_j \in W$. Recall that:

$$s_0 = t^{\theta^\vee/2} r_\theta t^{-\theta^\vee/2} = t^\varpi r_\theta t^{-\varpi},$$

where $\theta = x_1 - x_n$ is the highest root, since $\langle \theta, \theta^\vee/2 \rangle = \langle \theta, \varpi \rangle = 1$. Expressing this in terms of root subgroups $K_{(0)} \subset LK$ and $K_\theta \subset K \subset LK$, we have:

$$K_{(0)} = K_{(-\theta, 1)} = t^\varpi K_\theta t^{-\varpi}.$$

We may write:

$$\begin{aligned} t^{-\varpi} &= t^{-\pi(e_1)} = t^{\pi(e_2 + e_3 + \cdots + e_n)} \\ &= t^{u_1(\varpi)} t^{u_2(\varpi)} \cdots t^{u_{n-1}(\varpi)} \\ &= u_1 t^\varpi u_1^{-1} u_2 t^\varpi u_2^{-1} \cdots u_{n-1} t^\varpi u_{n-1}^{-1}, \end{aligned}$$

where:

$$u_i = s_i s_{i-1} \cdots s_1 = (i+1, i, \dots, 2, 1) \in W = S_n$$

is an $(i+1)$ -cycle with $u_i(1) = i+1$.

Now we can express the Schubert class $\Omega K_v \subset \Omega K$ in terms of the map Φ as follows:

$$\begin{aligned} \Omega K_v &= (K_{w_1} t^\varpi K_\theta t^{-\varpi}) \cdots (K_{w_r} t^\varpi K_\theta t^{-\varpi}) / T \\ &= (K_{w_1} t^\varpi K_\theta u_1 t^\varpi u_1 u_2 t^\varpi u_2^{-1} \cdots u_{n-1} t^\varpi u_{n-1}^{-1}) \cdots \\ &= \{k_1 t^\varpi k_1^{-1} \cdot (k_1 k'_1 u_1) t^\varpi (k_1 k'_1 u_1)^{-1} \cdot (k_1 k'_1 u_2) t^\varpi (k_1 k'_1 u_2)^{-1} \cdots\} \\ &= \{\Phi(k_1) \cdot \Phi(k_1 k'_1 u_1) \cdots \Phi(k_1 k'_1 u_r) \cdot \Phi(k_1 k'_1 k_2) \cdot \Phi(k_1 k'_1 k_2 k'_2 u_1) \cdots\}, \end{aligned}$$

where k_j runs over K_{w_j} and k'_j runs over K_θ for $j = 1, \dots, r$. Hence we define the submanifold $Y \subset X^m = (K/K^\varpi)^m$, where $m = nr$, as:

$$Y := \left\{ \begin{array}{l} (k_{(1)}, k'_{(1)} u_1, \dots, k'_{(1)} u_{n-1}, \\ k_{(2)}, k'_{(2)} u_1, \dots, k'_{(2)} u_{n-1}, \\ \vdots \\ k_{(r)}, k'_{(r)} u_1, \dots, k'_{(r)} u_{n-1}) \end{array} \left| \begin{array}{l} k_{(j)} := k_1 k'_1 \cdots k_{j-1} k'_{j-1} k_j, \quad k'_{(j)} := k_{(j)} k'_j \\ k_1 \in K_{w_1}, \dots, k_r \in K_{w_r}, \quad k'_1, \dots, k'_r \in K_\theta \end{array} \right. \right\},$$

so that $\Phi^m(Y) = \widehat{\Phi^m}(Y) = \Omega K_v = \Omega K_\lambda$, and $\Psi_{**}[Y] = [\Omega K_\lambda]$. Thus, we have reduced our problem to evaluating the fundamental class $[Y]$ as a polynomial in $\mathbb{Q}[h_1, \dots, h_{n-1}]$.

We remark that Y is an embedding of a Bott-Samelson variety:

$$Y \cong K_{w_1} \overset{T}{\times} K_\theta \overset{T}{\times} \cdots \overset{T}{\times} K_{w_r} \overset{T}{\times} K_\theta / T \cong \Omega K_{w_1, s_0, \dots, w_r, s_0}.$$

Indeed, we can express the Bott-Samelson variety of ΩK purely in terms $K_\alpha \subset K$ by virtue of the isomorphism of $(T \times T)$ -spaces $K_{(0)} \cong K_\theta$.

3.5. Demazure operations on multi-projective space. We consider the multi-projective space X^m , where $X = K/K^\varpi \cong \mathbb{P}^{n-1}$ and $m = nr$. For a root α of W , we recall the geometric Demazure operation $D_\alpha = D_\alpha^L$ which takes a T -invariant submanifold $Z \subset \tilde{X}^m$ to the new T -invariant submanifold:

$$D_\alpha(Z) := K_\alpha \cdot Z \subset \tilde{X}^m.$$

(Because we will use only left operations, we omit the superscript L .)

For a finite Weyl group element $w = s_{i_1} \circ \cdots \circ s_{i_\ell} \in W$, we can define $D_w := D_{(i_1)} \cdots D_{(i_\ell)}$, which is independent of the reduced decomposition. (Once more, $D_\alpha \neq D_{r_\alpha}$.) Also we recall the base point $\text{pt} = \text{id } K^\varpi \in X$ and the T -fixed points $u_i = u_i \cdot \text{pt} \in X$ for $i = 0, \dots, n-1$. Also, we abuse notation so that $w^{(k)} \in W^k$ can mean either a T -fixed point of X^k or an inclusion map $X^m \xrightarrow{\sim} w^{(k)} \times X^m \subset X^{k+m}$, and similarly with pt .

Now let us use these operations to construct the manifold $Y \subset X^m$ associated to the reduced factorization $v = w_1 \circ s_0 \circ \cdots \circ w_r \circ s_0$ in the previous section. Denote $u^{(n-1)} := (u_1, \dots, u_{n-1}) \in W^{n-1}$. Then we clearly have:

$$Y = D_{w_1} \text{pt } D_\theta u^{(n-1)} D_{w_2} \text{pt } D_\theta u^{(n-1)} \cdots D_{w_r} \text{pt } D_\theta u^{(n-1)}.$$

We proceed to interpret in polynomial terms each of these operations.

Consider the equivariant cohomology $H_T^*(\tilde{X}^m) := H^*(ET \times^T \tilde{X}^m)$. By the same argument as for the case $m = 1$, we have:

$$ET \times^T \tilde{X}^m \cong \overbrace{BT \times_{BK} \cdots \times_{BK} BT}^{m \text{ factors}} \times_{BK} BT.$$

Taking $H_T^*(\text{pt}) \cong S \cong \mathbb{Q}[y_1, \dots, y_n]/(y_1 + \cdots + y_n)$, we get:

$$\begin{aligned} H_T^*(\tilde{X}^m) &\cong S \otimes_{S^W} S \otimes_{S^W} \cdots \otimes_{S^W} S \\ &\cong \frac{\mathbb{Q}[x^{(1)}, x^{(2)}, \dots, x^{(m)}, y]}{\left(\begin{array}{c} y_1 + \cdots + y_n = 0 \\ h(x^{(1)}) = \cdots = h(x^{(m)}) = h(y), \forall h \in S^W \end{array} \right)}, \end{aligned}$$

where $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})$ and $y = (y_1, \dots, y_n)$, and the dimension grading becomes $\dim(x_j^{(k)}) = \dim(y_j) = 2$. We also have:

$$\begin{aligned} H_T^*(X^m) &\cong S^\varpi \otimes_{S^W} S^\varpi \otimes_{S^W} \cdots \otimes_{S^W} S^\varpi \otimes_{S^W} S \\ &\cong \frac{\mathbb{Q}[\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_m, y]}{\left(\begin{array}{c} y_1 + \cdots + y_n \\ \sum_{j=0}^n (-1)^j \tilde{x}_i^j e_{n-j}(y), i = 1, \dots, m \end{array} \right)}, \end{aligned}$$

where $\tilde{x}_k := x_1^{(k)}$.

The (left) geometric Demazure operation D_α induces via Poincare duality the cohomology operation:

$$\begin{aligned} \partial_\alpha^y : H_T^{2\ell}(\tilde{X}^m) &\rightarrow H_T^{2\ell-2}(\tilde{X}^m) \\ f(x, y) &\mapsto (-1) \frac{f(x, y) - f(x, r_\alpha y)}{y_i - y_j}, \end{aligned}$$

where $\alpha = y_i - y_j$; and we again have ∂_w^y for $w \in W$.

Recall the top-degree Schubert polynomial of $H_T^*(X) = S^\varpi \otimes_{S^W} S$, representing the basepoint of X :

$$[\text{pt} \in X]_T = \mathfrak{S}_{u_{n-1}}(\tilde{x}, y_1, \dots, y_n) = \prod_{j=1}^{n-1} (\tilde{x} - y_j),$$

as well as the other T -fixed points:

$$[u_i \in X]_T = \mathfrak{S}_{u_{n-1}}(\tilde{x}, u_i^*(y)) = \prod_{j \neq i+1} (\tilde{x} - y_j).$$

The embedding $u^{(n-1)} : X^k \hookrightarrow X^{k+n-1}$ induces the map:

$$\begin{aligned} [u^{(n-1)}]_T f(\tilde{x}_1, \dots, \tilde{x}_k, y) &:= \\ \mathfrak{S}_{u_{n-1}}(\tilde{x}_1, u_1^*(y)) \cdots \mathfrak{S}_{u_{n-1}}(\tilde{x}_{n-1}, u_{n-1}^*(y)) &f(\tilde{x}_n, \dots, \tilde{x}_{k+n-1}, y). \end{aligned}$$

We also have the fiber mapping $\perp : H_T^*(X) \rightarrow H^*(X)$, $f(\tilde{x}, y) \mapsto f(\tilde{x}, 0)$.

Now we write:

$$[Y] = \perp \partial_{w_1}^y [\text{pt}]_T \partial_{-\theta}^y [u^{(n-1)}]_T \partial_{w_2}^y [\text{pt}]^T \partial_{-\theta} [u^{(n-1)}]_T \cdots \partial_{w_r}^y [\text{pt}]_T \partial_{-\theta}^y [u^{(n-1)}]_T.$$

This is a cohomology class:

$$[Y] \in H^{2m(n-1)-2\ell}(X^m) \cong \left[\frac{\mathbb{Q}[\tilde{x}_1, \dots, \tilde{x}_m]}{(\tilde{x}_1^n, \dots, \tilde{x}_m^n)} \right]_{\dim=2m(n-1)-2\ell}$$

where $\ell = \ell(m^\lambda) := r + \ell(w_1) + \cdots + \ell(w_r)$ and $\dim(\tilde{x}_j) = 2$. This is Poincare dual to the fundamental class:

$$[Y] \in H_{2\ell}(X^m) \subset \text{Span}_{\mathbb{Q}}(1, h_1, \dots, h_{n-1})^{\otimes m}.$$

Symmetrizing induces the affine Schubert polynomial:

$$\mathfrak{S}_\lambda = \text{Sym}[Y] \in \text{Sym}^m \text{Span}_{\mathbb{Q}}(1, h_1, \dots, h_{n-1}) = \mathbb{Q}[h_1, \dots, h_{n-1}]_{\deg \leq m},$$

This has $\dim(\mathfrak{S}_\lambda) = 2\ell$, where the dimension grading is defined by $\dim(h_i) = 2i$; and $\deg(\mathfrak{S}_\lambda) \leq m = nr$, where \deg means the usual grading $\deg(h_i) = 1$.

3.6. Combinatorics of Schubert classes. In this section we collect the previous arguments into a self-contained combinatorial recipe to construct our affine Schubert polynomials:

THEOREM B: *For any $\lambda \in Q^\vee$, the polynomial*

$$\widehat{\mathfrak{S}}_\lambda \in \mathbb{Q}[h_1, \dots, h_{n-1}] = SH_*(X)$$

*defined by the steps (1)–(6) below satisfies $\Psi_{**}(\widehat{\mathfrak{S}}_\lambda) = [\Omega K_\lambda] \in H_*(\Omega K)$.*

(1) We have the affine Weyl group of periodic permutations of \mathbb{Z} :

$$\widehat{W} = \{\pi \in S_{\mathbb{Z}} \mid \pi(i+n) = \pi(i) + n\},$$

which is the semi-direct product of the finite Weyl group $W = S_n$, and the root lattice:

$$Q^\vee = \left\{ t^\lambda \mid \begin{array}{l} \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \\ \lambda_1 + \cdots + \lambda_n = 0 \end{array} \right\},$$

where $t^\lambda(i) := i + n\lambda_{(i \bmod n)}$. This is a Coxeter group generated by the simple reflections $s_i = (i, i+1) \in W$ for $i = 1, \dots, n-1$, and by $s_0 = r_\theta t^\theta$, where $r_\theta = (1, n)$ and $\theta = (1, 0, \dots, 0, -1)$, the highest coroot.

The cosets \widehat{W}/W have minimal representatives $m^\lambda = \min(t^\lambda W)$ for $\lambda \in Q^\vee$. We take a reduced factorization $m^\lambda = w_1 \circ s_0 \circ \cdots \circ w_r \circ s_0$ into finite Weyl group elements $w_1, \dots, w_r \in W$ separated by the affine simple reflection $s_0 \in \widehat{W}$.

- (2) We let $m = rn$, and we work with $m + n$ variables $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_m)$ and $y = (y_1, \dots, y_n)$. We define the ring:

$$\frac{\mathbb{Q}[\tilde{x}, y]}{\mathfrak{J}^\varpi} := \frac{\mathbb{Q}[\tilde{x}_1, \dots, \tilde{x}_m, y_1, \dots, y_n]}{\left(\begin{array}{l} y_1 + \dots + y_n = 0 \\ \sum_{j=0}^n (-1)^{n-j} \tilde{x}_k^j e_{n-j}(y), \quad k = 1, \dots, m \end{array} \right)},$$

where $e_i(y)$ denotes an elementary symmetric polynomial. The given polynomials are a Grobner basis of the ideal under the term order $y_1 < \dots < y_n < \tilde{x}_1 < \dots < \tilde{x}_m$.

- (3) For a root $\alpha = y_i - y_j$, we have the divided difference operator:

$$\partial_\alpha^y f(\tilde{x}, y) := \frac{f(\tilde{x}, y) - f(\tilde{x}, r_\alpha(y))}{y_i - y_j},$$

where r_α switches the variables y_i and y_j . Here α need not be a simple root: for example, it may be the longest root $\theta = y_1 - y_n$. For $w = s_{i_1} \circ \dots \circ s_{i_p} \in W$ a reduced factorization into simple reflections of W , we define $\partial_w^y := \partial_{\alpha_{i_1}}^y \dots \partial_{\alpha_{i_p}}^y$.

- (4) For $i = 1, \dots, n$, we have the twisted point-class polynomial:

$$\mathfrak{P}_i^{(k)} := \prod_{\substack{j=1 \\ j \neq i}}^n (\tilde{x}_k - y_j),$$

and for an interval of $n-1$ integers

$$[a+1, a+n-1] = \{a+1, a+2, \dots, a+n-1\},$$

we define:

$$\mathfrak{P}^{[a+1, a+n-1]} := \mathfrak{P}_{n-1}^{(a+1)} \mathfrak{P}_{n-2}^{(a+2)} \dots \mathfrak{P}_1^{(a+n-1)}.$$

- (5) To the factorization $m^\lambda = w_1 \circ s_0 \circ \dots \circ w_r \circ s_0$ we associate a polynomial $\tilde{\mathfrak{S}}(\tilde{x}, y) \in \mathbb{Q}[\tilde{x}, y]/\mathfrak{J}^\varpi$ by repeatedly applying the divided difference operators and multiplying by point-class polynomials:

$$\begin{aligned} \tilde{\mathfrak{S}}(\tilde{x}, y) := & \pm \partial_{w_1}^y \mathfrak{P}_n^{(1)} \partial_\theta^y \mathfrak{P}^{[2, n]} \partial_{w_2}^y \mathfrak{P}_n^{(n+1)} \partial_\theta^y \mathfrak{P}^{[n+2, 2n]} \dots \\ & \dots \partial_{w_r}^y \mathfrak{P}_n^{(rn-n+1)} \partial_\theta^y \mathfrak{P}^{[rn-n+2, rn]}, \end{aligned}$$

where the sign $\pm = (-1)^{\ell(w_1) + \dots + \ell(w_r)}$ takes care of orientations.

We specialize our polynomial to $y = 0$:

$$\tilde{\mathfrak{S}}(x) := \tilde{\mathfrak{S}}(x, 0) \in \frac{\mathbb{Q}[\tilde{x}]}{\mathfrak{J}_0^\varpi} := \frac{\mathbb{Q}[\tilde{x}_1, \dots, \tilde{x}_m]}{(\tilde{x}_1^n, \dots, \tilde{x}_m^n)}.$$

Note that $\deg(\tilde{\mathfrak{S}}) = m(n-1) - \ell(m^\lambda)$.

- (6) We consider $\mathbb{Q}[\tilde{x}]/\tilde{\mathfrak{J}}_0^\varpi$ as a \mathbb{Q} -vector space with a basis of monomials $\tilde{x}_1^{i_1} \cdots \tilde{x}_m^{i_m}$ with $0 \leq i_k \leq n-1$. Now we take the \mathbb{Q} -linear mapping:

$$\begin{aligned} \text{Sym} : \quad \mathbb{Q}[\tilde{x}]/\tilde{\mathfrak{J}}_0 &\longrightarrow \mathbb{Q}[h_1, \dots, h_{n-1}] \\ \tilde{x}_1^{i_1} \cdots \tilde{x}_m^{i_m} &\longmapsto h_{n-i_1-1} \cdots h_{n-i_m-1}, \end{aligned}$$

where $h_0 := 1$. The mapping $\tilde{x}_k^i \mapsto \tilde{x}^i$ corresponds to symmetrizing the tensor product, and the mapping $\tilde{x}^i \mapsto h_{n-i-1}$ undoes Poincare duality, turning a cohomology class into a homology class.

Finally, we define:

$$\widehat{\mathfrak{S}}_\lambda(h) := \text{Sym} \tilde{\mathfrak{S}}(x).$$

This is a polynomial with degree $\leq rn$ and with $\dim_{\mathbb{R}} \widehat{\mathfrak{S}}_\lambda(h) = \ell(m^\lambda)$, where $\dim_{\mathbb{R}}(h_i) := 2i$. It is independent of the choice of reduced factorization for m^λ .

3.7. Affine Schubert polynomials for SL_3 . Let us first apply the above recipe in the case $n = 3$. We can deduce from Theorem A that all Schubert classes are products of three “prime” classes:

$$\widehat{\mathfrak{S}}_\rho, \widehat{\mathfrak{S}}_{-\varpi_1}, \widehat{\mathfrak{S}}_{-\varpi_2}.$$

Here we have $\rho := \varpi_1 + \varpi_2$, and we write ϖ_i instead of ϖ_i^\vee for legibility. Also, for $\lambda \in P^\vee$, we denote $\widehat{\mathfrak{S}}_\lambda := \widehat{\mathfrak{S}}_{\hat{\lambda}}$ where $\hat{\lambda} \in Q^\vee$ is defined by $m^\lambda = \sigma_\lambda m^{\hat{\lambda}}$ with $\ell(\sigma_\lambda) = 0$. We have:

$$\widetilde{W} = \langle s_0, s_1, s_2, \sigma \rangle \quad \text{where} \quad \sigma s_i \sigma^{-1} = s_{(i+1 \bmod 3)}.$$

To simplify notation further, we will write x_k instead of \tilde{x}_k . We have:

$$\mathfrak{P}_i^{(k)} := (x_k - y_1)(x_k - y_2)(x_k - y_3) / (x_k - y_i).$$

We compute as follows:

- $\lambda = \rho = \varpi_1 + \varpi_2$, $m^\lambda = s_0$, $r = 1$, $m = 3$.
 - $\tilde{\mathfrak{S}}(x, y) := \mathfrak{P}_3^{(1)} \partial_{13}^y \mathfrak{P}_2^{(2)} \mathfrak{P}_1^{(3)}$
 $= (x_1 - y_1)(x_1 - y_2)(x_3 - y_2)(x_2 - y_3)(x_2 - y_1)$
 - $\tilde{\mathfrak{S}}(x) = x_1^2 x_2^2 x_3$, $\widehat{\mathfrak{S}}(h) = h_1$.
- $\lambda = -\varpi_1$, $m^\lambda = \sigma^2 s_2 s_0$, $r = 1$, $m = 3$.
 - $\tilde{\mathfrak{S}}(x, y) := \partial_{32} \mathfrak{P}_3^{(1)} \partial_{13}^y \mathfrak{P}_2^{(2)} \mathfrak{P}_1^{(3)}$
 $= (x_1 - y_2)(x_2 - y_1)(x_1 x_2 - x_1 x_3 + x_2 x_3 - x_2 y_2 - x_2 y_3 + y_2 y_3)$
 - $\tilde{\mathfrak{S}}(x) = x_1^2 x_2^2 - x_1^2 x_2 x_3 + x_2 x_2^2 x_3$
 $\widehat{\mathfrak{S}}(h) = h_2 - h_1^2 + h_1^2 = h_2$.

- $\lambda = -\varpi_2$, $m^\lambda = \sigma s_1 s_0$, $r = 1$, $m = 3$.
 - $\tilde{\mathfrak{S}}(x, y) := \partial_{21} \mathfrak{P}_3^{(1)} \partial_{13}^y \mathfrak{P}_2^{(2)} \mathfrak{P}_1^{(3)}$
 $= -(x_2 - x_3)(x_2 - y_3)(x_1 - y_1)(x_1 - y_2)$
 - $\tilde{\mathfrak{S}}(x) = x_1^2 x_2 x_3 - x_1^2 x_2^2$, $\hat{\mathfrak{S}}(h) = h_1^2 - h_2$.
- $\lambda = -2\varpi_2$, $m^\lambda = \sigma^2 s_0 s_2 s_1 s_0$, $r = 2$, $m = 6$.
 - $\tilde{\mathfrak{S}}(x, y) := \mathfrak{P}_3^{(1)} \partial_{13}^y \mathfrak{P}_2^{(2)} \mathfrak{P}_1^{(3)} \partial_{32} \partial_{21} \mathfrak{P}_3^{(4)} \partial_{13}^y \mathfrak{P}_2^{(5)} \mathfrak{P}_1^{(6)}$
 - $\tilde{\mathfrak{S}}(x) = -x_1^2 x_2^2 x_3 (x_3 - x_4)(x_5 - x_6)(x_4 - x_5)$
 $\hat{\mathfrak{S}}(h) = (h_1^2 - h_2)^2$, consistent with Theorem A.

3.8. Affine Schubert polynomials for SL_4 . For $n = 4$, we list the $\lambda \in P^\vee$ corresponding to the “prime” classes $\mathfrak{S}_\lambda \in \mathbb{Q}[h_1, h_2, h_3]$.

- $\lambda = -\varpi_1$, $m^\lambda = \sigma^3 s_2 s_3 s_0$, $\mathfrak{S}_\lambda = h_3$.
- $\lambda = -\varpi_2$, $m^\lambda = \sigma^2 s_0 s_3 s_1 s_0$, $\mathfrak{S}_\lambda = h_2^2 - h_1 h_3$.
- $\lambda = -\varpi_3$, $m^\lambda = \sigma s_2 s_1 s_0$, $\mathfrak{S}_\lambda = h_1^3 - 2h_1 h_2 + h_3$.
- $\lambda = \rho = \varpi_1 + \varpi_2 + \varpi_3$, $m^\lambda = \sigma^2 s_2 s_1 s_3 s_0$, $\mathfrak{S}_\lambda = h_2(h_1^2 - h_2)$.
- $m^\lambda = \sigma^2 s_1 s_3 s_0$, $\mathfrak{S}_\lambda = h_1 h_2 - h_3$.
- $m^\lambda = \sigma^2 s_3 s_0$, $\mathfrak{S}_\lambda = h_2$.
- $m^\lambda = \sigma^2 s_1 s_0$, $\mathfrak{S}_\lambda = h_1^2 - h_2$.
- $m^\lambda = \sigma^2 s_0$, $\mathfrak{S}_\lambda = h_1$.

Notice the bit of mysterious extra factorization for \mathfrak{S}_ρ .

3.9. Calculations for higher SL_n . For general n , the “prime” Schubert classes include $\mathfrak{S}_{-\varpi_i}$, the classes corresponding to the fundamental anti-dominant translations $t^{-\varpi_i} = m^{-\varpi_i}$. (We again identify ϖ_i with ϖ_i^\vee , etc.) We can decompose this as:

$$t^{-\varpi_i} = (s_i s_{i-1} \cdots s_2 s_1 \sigma)^{n-i} = w_i w_0 \sigma^{-i},$$

where $w_i = \max W^{\varpi_i}$, the longest element of a maximal parabolic, and σ is the automorphism of the fundamental alcove with:

$$\sigma s_i \sigma^{-1} = s_{i+1} \quad , \quad \text{where } s_n := s_0 .$$

Now, $\lambda = -\varpi_i \notin Q^\vee$, so \mathfrak{S}_λ means $\mathfrak{S}_{\hat{\lambda}}$, where $\hat{\lambda} \in Q^\vee$ is defined by $m^{\hat{\lambda}} = \sigma^i t^{-\varpi_i}$. Indeed, Shimozone has calculated that, letting $k := \min(i, n-i)$,

we have:

$$\widehat{\lambda} = k\alpha_i + \sum_{j=1}^{k-1} (k-j)(\alpha_{i-j} + \alpha_{i+j}).$$

For example for $n = 6$, we have:

$$\begin{aligned} \widehat{-\varpi_1} &= \alpha_1 & \widehat{-\varpi_2} &= \alpha_1 + 2\alpha_2 + \alpha_3 & \widehat{-\varpi_3} &= \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 \\ \widehat{-\varpi_5} &= \alpha_5 & \widehat{-\varpi_4} &= \alpha_3 + 2\alpha_4 + \alpha_5 \end{aligned}$$

The other prime classes are \mathfrak{S}_λ for $\lambda \in Q^\vee$ with $(m^\lambda)^{-1}$ lying inside the fundamental box, as described in the Appendix. The fundamental box has maximal alcove

$$m^\rho = t^\rho w_0 \in \widetilde{W},$$

where $\rho = \sum_{i=1}^{n-1} \varpi_i$, which is also the half-sum of the positive coroots. This element is an involution:

$$(m^\rho)^{-1} = w_0 t^\rho = t^{-w_0(\rho)} w_0 = m^\rho.$$

We can decompose this as $m^\rho = \iota \hat{m}^\rho$, where $\hat{m}^\rho \in \widehat{W}$ and:

$$\iota = \begin{cases} 1 & \text{for } n \text{ odd} \\ \sigma^{n/2} & \text{for } n \text{ even.} \end{cases}$$

The alcoves in the fundamental box are the right weak order interval $[1, \hat{m}]_R$, and the corresponding prime classes are:

$$[1, \hat{m}^{-1}]_L = [1, \iota \hat{m} \iota]_L = \left\{ w \in \widehat{W} \mid \iota \hat{m} \iota = u \circ w \text{ for some } u \right\}.$$

4. APPENDIX: WEAK ORDER ON AN AFFINE WEYL GROUP

This note addresses the following question concerning optimal factorization in an affine Weyl group.

4.1. Question. Given an affine Weyl group element $x \in \widehat{W}$, minimal in its coset xW with respect to the finite Weyl group W , find the maximal dominant coweight λ such that there exists a factorization $x = y \cdot t_{-\lambda}$ with $\ell(x) = \ell(y) + \ell(t_{-\lambda})$, where $y \in \widetilde{W}$, the extended affine Weyl group, and $t_{-\lambda} \in \widetilde{W}$ is an anti-dominant translation.

4.2. Answer. Let $\varpi_1^\vee, \dots, \varpi_n^\vee$ be the fundamental coweights of W . For a vector $\gamma = c_1 \varpi_1^\vee + \dots + c_n \varpi_n^\vee$ with $c_i \in \mathbb{R}$, define its floor as the integral coweight:

$$\lfloor \gamma \rfloor := \lfloor c_1 \rfloor \varpi_1^\vee + \dots + \lfloor c_n \rfloor \varpi_n^\vee,$$

where $\lfloor c \rfloor$ denotes the integer floor of a real number.

Choose a tiny positive vector $\epsilon := \epsilon_1 \varpi_1^\vee + \dots + \epsilon_n \varpi_n^\vee$, where $\epsilon_i > 0$ are sufficiently small positive real numbers. Then the dominant coweight asked for is: $\lambda = \lfloor x^{-1}(\epsilon) \rfloor$, which is independent of the choice of ϵ .

4.3. Examples.

a. For $x = wt_\mu$, where $w \in W$ and μ is a dominant coweight, this gives:

$$\lambda = \mu - \sum_{i \in D(w^{-1})} \varpi_i^\vee,$$

where $D(w^{-1}) := \{i \in [1, n] \mid w^{-1}(\alpha_i) < 0\}$, the descent set of w^{-1} .

- b. Let \widehat{W} be of type $A_2^{(1)}$ and $x = s_1 s_0$. We have $s_1(\varpi_1^\vee) = -\varpi_1^\vee + \varpi_2^\vee$, $s_2(\varpi_2^\vee) = \varpi_1^\vee - \varpi_2^\vee$, and $s_i(\varpi_j^\vee) = \varpi_j^\vee$ for $1 \leq i \neq j \leq n$. Also, $s_0 = t_{\theta^\vee} r_\theta$, where $\theta^\vee = \varpi_1^\vee + \varpi_2^\vee$ and $r_\theta(\varpi_1^\vee) = -\varpi_2^\vee$, $r_\theta(\varpi_2^\vee) = -\varpi_1^\vee$. Then $\epsilon = \epsilon_1 \varpi_1^\vee + \epsilon_2 \varpi_2^\vee$, and $x^{-1}(\epsilon) = (1 - \epsilon_1 - \epsilon_2) \varpi_1^\vee + (1 + \epsilon_2) \varpi_2^\vee$, so $\lambda = 0 \varpi_1^\vee + 1 \varpi_2^\vee = \varpi_2^\vee$.
- c. Let \widehat{W} be of type $C_2^{(1)}$ with Dynkin diagram $0 \Rightarrow 1 \Leftarrow 2$, and $x = s_0 s_1 s_0$. We have $s_1(\varpi_1^\vee) = -\varpi_1^\vee + 2\varpi_2^\vee$, $s_2(\varpi_2^\vee) = \varpi_1^\vee - \varpi_2^\vee$, and $s_i(\varpi_j^\vee) = \varpi_j^\vee$ for $1 \leq i \neq j \leq n$. Also, $s_0 = t_{\theta^\vee} r_\theta$, where $\theta^\vee = \varpi_1^\vee$ and $r_\theta(\varpi_1^\vee) = -\varpi_1^\vee$, $r_\theta(\varpi_2^\vee) = -\varpi_1^\vee + \varpi_2^\vee$. Then $\epsilon = \epsilon_1 \varpi_1^\vee + \epsilon_2 \varpi_2^\vee$, and $x^{-1}(\epsilon) = \epsilon_1 \varpi_1^\vee + (2 - 2\epsilon_1 - \epsilon_2) \varpi_2^\vee$, so $\lambda = 0 \varpi_1^\vee + 1 \varpi_2^\vee = \varpi_2^\vee$.

4.4. Root systems and alcoves. We proceed to prove the above answer, as well as record some general facts for reference (cf. [H], [S]).

Let \mathbb{R}^n and $(\mathbb{R}^n)^*$ be dual vector spaces. A positive definite inner product allows one to identify \mathbb{R}^n and $(\mathbb{R}^n)^*$, but we will use only the natural pairing $\langle \cdot, \cdot \rangle$ between $(\mathbb{R}^n)^*$ and \mathbb{R}^n . We define a root system by a set $\Delta \subset (\mathbb{R}^n)^*$ of root vectors and a set $\Delta^\vee \subset \mathbb{R}^n$ of coroot vectors. We have the bases of simple roots $\alpha_1, \dots, \alpha_n \in (\mathbb{R}^n)^*$ and simple coroots $\alpha_1^\vee, \dots, \alpha_n^\vee \in \mathbb{R}^n$, together with their dual bases, the fundamental weights $\varpi_1, \dots, \varpi_n \in (\mathbb{R}^n)^*$ and fundamental coweights $\varpi_1^\vee, \dots, \varpi_n^\vee \in \mathbb{R}^n$. For $\alpha \in \Delta$, we have the reflection $r_\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mu \mapsto \mu - \langle \alpha, \mu \rangle \alpha^\vee$. The simple reflections s_1, \dots, s_n generate the finite Coxeter group W .

For a vector $\lambda \in \mathbb{R}^n$ we have the translation $t_\lambda : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $\mu \mapsto \mu + \lambda$, so that $w t_\lambda w^{-1} = t_{w(\lambda)}$ for $w \in W$. We define the affine Weyl group $\widehat{W} = W \ltimes t_{Q^\vee}$ and the extended affine Weyl group $\widetilde{W} = W \ltimes t_{P^\vee}$, where $Q^\vee = \mathbb{Z}\alpha_1^\vee \oplus \dots \oplus \mathbb{Z}\alpha_n^\vee$ is the coroot lattice and $P^\vee = \mathbb{Z}\varpi_1^\vee \oplus \dots \oplus \mathbb{Z}\varpi_n^\vee$ is the coweight lattice. As a Coxeter group, \widehat{W} has generators s_0, s_1, \dots, s_n , where $s_0 = t_{\theta^\vee} r_\theta = r_\theta t_{-\theta^\vee}$.

The affine reflections $r_{\alpha, k} \in \widehat{W}$ correspond to the hyperplanes

$$H_{\alpha, k} = \{\mu \mid \langle \alpha, \mu \rangle = k\}$$

for $\alpha \in \Delta_+$ and $k \in \mathbb{Z}$. The (closures of) connected components of $\mathbb{R}^n \setminus \{H_{\alpha, k}\}$ are called *alcoves*: they are simplices provided Δ is an irreducible root system. We define the fundamental alcove

$$\begin{aligned} A_0 &:= \{\mu \mid 0 \leq \langle \alpha, \mu \rangle \leq 1 \text{ for all } \alpha \in \Delta_+\} \\ &= \{\mu \mid \langle \alpha_1, \mu \rangle, \dots, \langle \alpha_n, \mu \rangle \geq 0, \langle \theta, \mu \rangle \leq 1\}. \end{aligned}$$

This is a fundamental domain of \widehat{W} acting on \mathbb{R}^n , and each alcove A corresponds to a unique $w \in \widehat{W}$ with $A = wA_0$. Henceforth we identify $w \in \widehat{W}$ with its alcove wA_0 .

Since the neighbors of A_0 are $s_i A_0$, any two neighboring alcoves A, A' are of the form $A = wA_0$ and $A' = ws_i A_0$ for some $i = 0, \dots, n$; and we label the common facet of A and A' with i . The length function on \widehat{W} (or on the set of alcoves) is defined as: $\ell(w) = \#$ hyperplanes separating w from 1.

The extended \widehat{W} is the symmetry group of the affine hyperplane arrangement $\{H_{\alpha,k}\}$, or of the set of alcoves, and the quotient $\widetilde{W}/\widehat{W}$ is isomorphic to the symmetry group of A_0 itself: $\sigma A_0 = A_0$ for each minimal coset representative $\sigma \in \widetilde{W}/\widehat{W}$. That is, $\widetilde{W} = \Sigma \ltimes \widehat{W}$, where $\Sigma := \{\sigma \in \widetilde{W} \mid \sigma(A_0) = A_0\}$. Every element $x \in \widetilde{W}$ can be uniquely factored as:

$$x = {}^\wedge x \sigma = \sigma x^\wedge$$

where $\sigma \in \Sigma$ and ${}^\wedge x, x^\wedge \in \widehat{W}$. We have $xA_0 = {}^\wedge x A_0$, and we define $\ell(x) := \ell({}^\wedge x) = \ell(x^\wedge)$. In particular, for each $\lambda \in P^\vee$ we have $t_\lambda = {}^\wedge t_\lambda \sigma$ with ${}^\wedge t_\lambda \in \widehat{W}$ and some $\sigma \in \widetilde{W}/\widehat{W}$.

We also have the fundamental chamber $C_0 = \{\mu \mid \langle \alpha_1, \mu \rangle, \dots, \langle \alpha_n, \mu \rangle \geq 0\}$, and a translation t_λ lies in C_0 exactly when λ is dominant. Furthermore, for λ a dominant coweight, wt_λ lies in the chamber wC_0 , and $t_{-\lambda}$ lies in $-C_0 = w_0 C_0$.

Lusztig has given the following length formula for $wt_\lambda \in \widetilde{W}$, where $w \in W$ and $\lambda \in P^\vee$:

$$\ell(wt_\lambda) = \sum_{\alpha \in \Delta_+} |\langle \lambda, \alpha \rangle| + \ell(w).$$

4.5. Strong and weak orders. There are three important orders on \widehat{W} and \widetilde{W} . The *strong* or *Bruhat order* \leq^B has covering relations $x \leq^B r_{\alpha,k} x$, where $r_{\alpha,k}$ is any reflection with $\ell(r_{\alpha,k} x) = \ell(x) + 1$. We have $x \leq^B y$ iff $x^{-1} \leq^B y^{-1}$.

In this note, we will mainly consider the weak orders: the *left weak order* \leq^L has covers $x \leq^L s_i x$, where s_i ($i = 0, \dots, n$) is a simple reflection with $\ell(s_i x) = \ell(x) + 1$. The analogous *right weak order* \leq^R has covers $x \leq^R x s_i$. Since $x \leq^L y$ iff $x^{-1} \leq^R y^{-1}$, the left and right orders are combinatorially equivalent. However, the right order is easier to picture in terms of alcoves: the covering relation $x \leq^R x s_i$ steps from an alcove $A = xA_0$ to its neighbor $A' = x s_i A_0$ through a facet labelled i , provided the facet separates A' from the fundamental A_0 .

These orders become very simple on dominant translations t_λ . That is, let λ, μ be dominant coweights. Then we have:

$$\begin{aligned} t_\lambda \stackrel{B}{\leq} t_\mu &\iff \mu - \lambda \in Q_+^\vee := \mathbb{N}\alpha_1^\vee \oplus \cdots \oplus \mathbb{N}\alpha_n^\vee \stackrel{\text{def}}{\iff} \lambda \stackrel{Q}{\leq} \mu, \\ {}^\wedge t_\lambda \stackrel{R}{\leq} {}^\wedge t_\mu &\iff \mu - \lambda \in P_+^\vee := \mathbb{N}\varpi_1^\vee \oplus \cdots \oplus \mathbb{N}\varpi_n^\vee \stackrel{\text{def}}{\iff} \lambda \stackrel{P}{\leq} \mu. \end{aligned}$$

In the second line we must use ${}^\wedge t$ instead of t , since we can only have $t_\mu \stackrel{R}{\leq} t_\lambda$ when $t_\mu \equiv t_\lambda \pmod{\widehat{W}}$, meaning $\mu - \lambda \in Q^\vee$.

When restricted to an orbit $W \cdot t^\lambda$ for $\lambda \in P_+^\vee$, the affine Bruhat order becomes the finite Bruhat order on the parabolic quotient W/W_λ . If $W_\lambda = \{1\}$, we have for $w, y \in W$:

$$wt_\lambda \stackrel{B}{\leq} yt_\lambda \iff w \stackrel{B}{\leq} y \implies w(\lambda) \stackrel{Q}{\geq} y(\lambda).$$

Warning: The converse of the last implication does *not* hold.¹⁰ To characterize the weak order on W , define the inversion set:

$$\text{Inv}(w) := \Delta_+ \cap w^{-1}\Delta_- = \{\alpha^{(1)}, \dots, \alpha^{(n)}\},$$

where $\alpha^{(k)} := s_{i_n} s_{i_{n-1}} \cdots s_{i_{k+1}}(\alpha_{i_k})$. Now we have: $y \stackrel{L}{\leq} w$ iff $\text{Inv}(y) \subset \text{Inv}(w)$, and $y \stackrel{R}{\leq} w$ iff $\text{Inv}(y^{-1}) \subset \text{Inv}(w^{-1})$. To define the strong and left weak orders on W/W_λ , we replace the cosets wW_λ by their shortest (or longest) representatives and take the orders induced from W .

The orders can be adapted to parabolic quotients of the affine Weyl group. Consider $\widehat{W}/W = \{t_\lambda W\}_{\lambda \in Q^\vee}$, the right cosets of the affine Weyl group by the finite Weyl group. Pictorially, each coset corresponds to a translate $t_\lambda P_0$ of the Weyl polytope $P_0 := \bigcup_{w \in W} wA_0$ by an element of the coroot lattice. Let $\min(xW)$, $\max(xW)$ denote the minimal and maximal element in the coset. We define: $\ell(xW) = \ell(\min(xW))$, and $xW \stackrel{L}{\leq} s_i xW$ whenever $\ell(s_i xW) = \ell(xW) + 1$. In fact,

$$xW \stackrel{L}{\leq} yW \iff \min(xW) \stackrel{L}{\leq} \min(yW) \iff \max(xW) \stackrel{L}{\leq} \max(yW).$$

The left cosets $Wx \in W \backslash \widehat{W}$ can best be pictured by their minimal representatives, the alcoves inside the fundamental chamber: $A \subset C_0$ with $A = \min(Wx) = t_\lambda w$, where $\lambda \in Q_+^\vee$ and $w \in W$ with w minimal in the coset $w \text{Stab}_W(\lambda)$. We define $Wx \stackrel{R}{\leq} Wy$ so that: $xW \stackrel{L}{\leq} yW$ iff $Wx^{-1} \stackrel{R}{\leq} Wy^{-1}$.

¹⁰ Example: $W = S_{n+1}$, $\lambda = \rho = (n+1, n, \dots, 2, 1) \in P_+^\vee \subset \mathbb{R}^{n+1}$. We have:

$$\begin{aligned} w \stackrel{B}{\leq} y &\iff \{w(1), \dots, w(i)\} \leq \{y(1), \dots, y(i)\} \text{ for } i = 1, \dots, n, \\ w(\rho) \stackrel{Q}{\geq} y(\rho) &\iff w(1) + \cdots + w(i) \leq y(1) + \cdots + y(i) \text{ for } i = 1, \dots, n. \end{aligned}$$

For $w = 2314$, $y = 4123$, the second condition holds but not the first.

The weak orders extend in an obvious way to \widetilde{W}/W and $W \setminus \widetilde{W}$. The Bruhat order on \widetilde{W}/W can be defined similarly, and I believe the minimal covers of $xW = vt_\lambda W$ for $\lambda \in P_+^\vee$ and $v \in W$ are:

$$vt_\lambda W \stackrel{B}{\leq} \begin{cases} r_\alpha v t_\lambda W & \text{if } \ell(r_\alpha v) = \ell(v) + 1 \\ r_\alpha v t_{\lambda - \alpha_i^\vee} W & \text{if } \ell(r_\alpha v) = \ell(v) - 1. \end{cases}$$

I do not know a good criterion for when $vt_\lambda \stackrel{B}{\leq} wt_\mu$.

4.6. Factorization and weak order. A *reduced factorization* is an expression $x = y \cdot z$, where $x = yz \in \widetilde{W}$ and $\ell(x) = \ell(y) + \ell(z)$. This is closely related to the weak order: for $x, y, z \in \widehat{\widetilde{W}}$,

$$x = y \cdot z \iff x \stackrel{L}{\leq} z \iff x \stackrel{R}{\leq} y.$$

For $x, y, z \in \widehat{\widetilde{W}}$, this becomes:

$$x = y \cdot z \iff x^\wedge \stackrel{L}{\leq} z^\wedge \iff {}^\wedge x \stackrel{R}{\leq} {}^\wedge y,$$

since if $x = y \cdot z$ with $y = {}^\wedge y \sigma$, $z = {}^\wedge z \sigma'$, then $x = {}^\wedge x \sigma \sigma'$ and ${}^\wedge x = {}^\wedge y \cdot (\sigma {}^\wedge z \sigma^{-1})$ with $\sigma {}^\wedge z \sigma^{-1} \in \widehat{\widetilde{W}}$. The converse means: if ${}^\wedge x \stackrel{R}{\leq} {}^\wedge y$, then for any $\sigma, \sigma'' \in \Sigma$, there exists $z \in \widehat{\widetilde{W}}$ such that ${}^\wedge x \sigma'' = {}^\wedge y \sigma \cdot z$. Similarly for x^\wedge .

Now we can rephrase the Question. Given $x \in \widehat{\widetilde{W}}$, minimal in its coset xW , we want the maximal $\lambda \in P_+^\vee$ such that for some $y \in \widehat{\widetilde{W}}$, we have: $x = y \cdot t_{-\lambda}$, meaning $x^{-1} = t_\lambda \cdot y^{-1}$, meaning ${}^\wedge t_\lambda \stackrel{R}{\leq} x^{-1}$.

4.7. Question reformulated: Given $x^{-1} \in \widehat{\widetilde{W}}$, minimal in its coset Wx^{-1} , find the $\stackrel{P}{\leq}$ -maximal coweight $\lambda \in P_+^\vee$ such that ${}^\wedge t_\lambda \stackrel{R}{\leq} x^{-1}$.

This has an obvious pictorial answer. The group t_{P^\vee} acts on \mathbb{R}^n with fundamental domain equal to the parallelpiped spanned by the fundamental coweights:

$$B_0 := \{\mu \mid 0 \leq \langle \alpha_i, \mu \rangle \leq 1 \text{ for } i = 1, \dots, n\}$$

We call B_0 the *fundamental box*, and its translates $B_\lambda := t_\lambda B_0$ cover $\mathbb{R}^n = \bigcup_{\lambda \in P^\vee} B_\lambda$. Each box is a union of N alcoves, where $N = |W| / [\widetilde{W} : \widehat{W}] = |W| / [P^\vee : Q^\vee]$. Furthermore, the fundamental chamber is the union of the dominant boxes: $C_0 = \bigcup_{\lambda \in P_+^\vee} B_\lambda$.

Now we have: ${}^\wedge t_\lambda \stackrel{R}{\leq} y$ iff $yA_0 \subset B_\lambda$. (The \Rightarrow is easy. The \Leftarrow is because ...) Hence, given $y = x^{-1} \in \widehat{\widetilde{W}}$ we have:

$$\max\{\mu \mid {}^\wedge t_\mu \stackrel{R}{\leq} y\} = \lambda \quad \text{where} \quad yA_0 \subset B_\lambda.$$

Now, if we take any point $\epsilon = \epsilon_1 \varpi_1^\vee + \dots + \epsilon_n \varpi_n^\vee$ in the interior of A_0 , we have:

$$yA_0 \subset B_\lambda \iff y(\epsilon) \in B_\lambda \iff \forall i : \langle \alpha_i, \lambda \rangle < \langle \alpha_i, y(\epsilon) \rangle < \langle \alpha_i, \lambda \rangle + 1.$$

Therefore, the answer to our question is exactly:

$$\lambda := \lfloor \langle \alpha_1, y(\epsilon) \rangle \rfloor \varpi_1^\vee + \cdots + \lfloor \langle \alpha_n, y(\epsilon) \rangle \rfloor \varpi_n^\vee,$$

which was to be shown.

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Peter Magyar
 magyar@math.msu.edu
 Dept of Math, Wells Hall
 Michigan State University
 East Lansing, MI 48823